

# Path integral

$$K(\underline{r}_f, \underline{r}_0, t) = \langle \underline{r}_f | e^{-\frac{i}{\hbar} \hat{H} t} | \underline{r}_0 \rangle = \int \mathcal{D}[\underline{r}] e^{\frac{i}{\hbar} S[\underline{r}]}$$

↑  
int. over trajectories  
from  $\underline{r}_0$  to  $\underline{r}_f$

seen so far:

• for small times

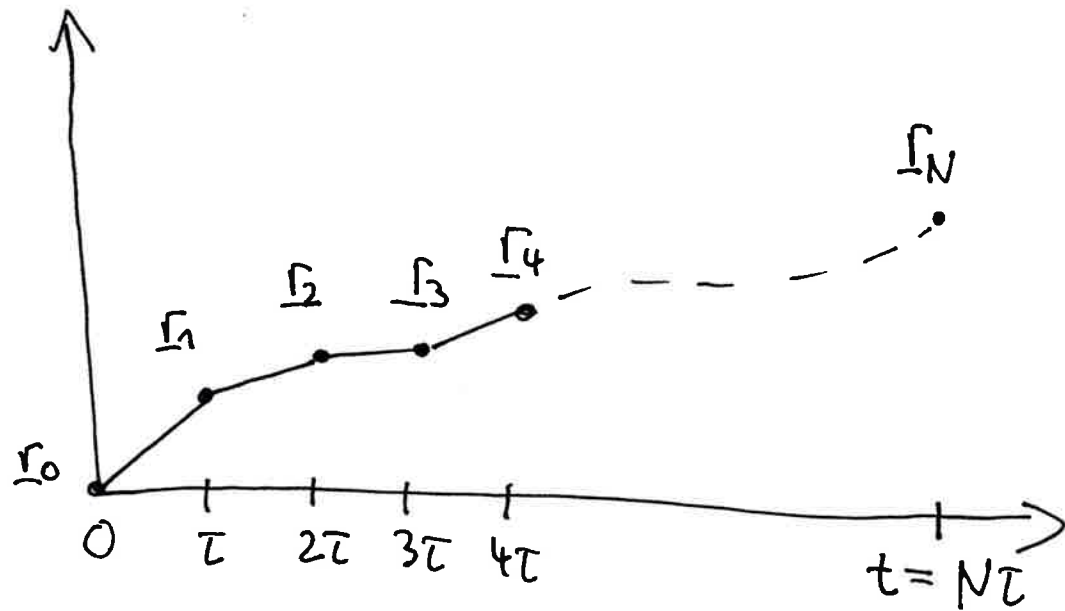
$$K(\underline{r}_f, \underline{r}_0, t) \approx \left( \frac{m}{2\pi i \hbar t} \right)^{n/2} \exp\left( \frac{i}{\hbar} \left[ \frac{1}{2} m \left( \frac{\underline{r}_f - \underline{r}_0}{t} \right)^2 - U(\underline{r}_0) \right] t \right)$$

• for arbitrary times

$$K(\underline{r}_f, \underline{r}_0, t) \approx \left( \frac{m N}{2\pi i \hbar t} \right)^{nN/2} \int d^n r_1 \dots \int d^n r_{N-1} \exp\left( \frac{i}{\hbar} \sum_{j=0}^{N-1} \left[ \frac{1}{2} m \left( \frac{\underline{r}_{j+1} - \underline{r}_j}{\tau} \right)^2 - U(\underline{r}_j) \right] \tau \right)$$

$$\left( \begin{array}{l} N \rightarrow \infty \\ \tau = \frac{t}{N} \end{array} \right)$$

interpret  $\underline{r}_j$  as position after time  $j\tau$



$$K(\underline{r}_f, \underline{r}_0, t) \simeq \left( \frac{mN}{2\pi i \hbar t} \right)^{nN/2} \int d^n r_1 \dots \int d^n r_{N-1} \exp\left( \frac{i}{\hbar} \sum_{j=0}^{N-1} \left[ \frac{1}{2} m \left( \frac{\underline{r}_{j+1} - \underline{r}_j}{\tau} \right)^2 - U(\underline{r}_j) \right] \tau \right)$$

good approx. for Lagrangian  
in time interval from  
 $j\tau$  to  $(j+1)\tau$

good approx. for action

$$\underline{N \rightarrow \infty}$$

approximation becomes better and better  
get integral over trajectories evaluated at all times

$$\sum_{j=0}^{N-1} \left[ \frac{1}{2} m \left( \frac{r_{j+1} - r_j}{\tau} \right)^2 - U(r_j) \right] \tau$$

$$\rightarrow \int_0^t L(r(t'), \dot{r}(t')) dt' = S[r]$$

$$K(r_f, r_0, t) = \int_{\substack{r(0) = r_0 \\ r(t) = r_f}} \mathcal{D}[r] e^{\frac{i}{\hbar} S[r]}$$

$$\text{with } \int \mathcal{D}[r] \dots = \lim_{N \rightarrow \infty} \left( \frac{m N}{2\pi i \hbar t} \right)^{nN/2} \int d^n r_1 \dots \int d^n r_{N-1} \dots$$

## Remarks

- **rigorous** treatment requires care with:  
integration measure, non-differentiable trajectories
- all trajectories needed
- action is stationary for trajectories satisfying  
eqs of motion

$$S[\Gamma_{cl} + \delta \underline{\rho}] = S[\Gamma_{cl}] + \mathcal{O}((\delta \underline{\rho})^2)$$

↑                    ↑  
satisfies        deviates  
e.o.m

traj. similar to classical ones give similar contribution  
important joint contribution

⇒ **classical trajectories dominate**

- if some  $\underline{\Gamma}$  are not possible, can have additional phases

- Hamiltonian mechanics version:

$$H = \underline{p} \cdot \dot{\underline{r}} - L \Rightarrow L = \underline{p} \cdot \dot{\underline{r}} - H$$

$$K(\underline{r}_f, \underline{r}_0, t) = \int \mathcal{D}[\underline{r}] \mathcal{D}[\underline{p}] \exp\left(\frac{i}{\hbar} \int_0^t (\underline{p}(t') \cdot \dot{\underline{r}}(t') - H(\underline{r}(t'), \underline{p}(t'))) dt'\right)$$

↑  
trajectories in phase space  
with  $\underline{r}(0) = \underline{r}_0, \underline{r}(t) = \underline{r}_f$

## 2.3 Example: Harmonic Oscillator

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$S = \int_0^t \left( \frac{1}{2} m \dot{x}(t')^2 - \frac{1}{2} m \omega^2 x(t')^2 \right) dt'$$

split  $x(t')$  as:

$$x(t') = \underset{\substack{\uparrow \\ \text{solution of} \\ \text{classical laws} \\ \text{of motion}}}{x_{cl}(t')} + \underset{\substack{\uparrow \\ \text{deviation} \\ \text{with} \\ \delta x(0) = \delta x(t) = 0}}{\delta x(t')}$$

action:

$$S[x] = \int_0^t \left( \frac{1}{2} m \dot{x}_{cl}^2(t') - \frac{1}{2} m \omega^2 x_{cl}^2(t') \right) dt'$$

+ linear terms (must vanish due to stationarity of action)

$$+ \int_0^t \left( \frac{1}{2} m \dot{\delta x}(t')^2 - \frac{1}{2} m \omega^2 \delta x(t')^2 \right) dt'$$

$$= S[x_{cl}] + S[\delta x]$$

determine  $x_{cl}$ :

$$\frac{\partial L}{\partial x} = \frac{d}{dt'} \frac{\partial L}{\partial \dot{x}}$$

$$-m\omega^2 x_{cl} = m \ddot{x}_{cl}$$

$$x_{cl}(t') = A \sin \omega t' + B \cos \omega t'$$

$$S[x_{cl}] = \frac{m\omega}{2} \left[ (A^2 - B^2) \frac{\sin 2\omega t}{2} + 2AB \frac{\cos 2\omega t - 1}{2} \right]$$

boundary conditions:

$$x_{cl}(0) = B \stackrel{!}{=} x_0$$

$$x_{cl}(t) = A \sin \omega t + B \cos \omega t \stackrel{!}{=} x_f \Rightarrow A = \frac{x_f - B \cos \omega t}{\sin \omega t}$$

$$S[x_{cl}] = \frac{m\omega}{2 \sin \omega t} \left( (x_0^2 + x_f^2) \cos \omega t - 2x_0 x_f \right)$$

Result so far

$$K(x_f, x_0, t) = \int \mathcal{D}[x] e^{\frac{i}{\hbar}(S[x_{cl}] + S[\delta x])}$$

would like  $\uparrow$   $\mathcal{D}[\delta x]$

$$\int \mathcal{D}[x] \dots = \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{\frac{N}{2}} \int dx_1 \dots dx_{N-1} \dots$$

$$\stackrel{\substack{\uparrow \\ x_j = x_{cl,j} + \delta x_j}}{=} \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{\frac{N}{2}} \int d\delta x_1 \dots d\delta x_{N-1} \dots$$

$$= \int \mathcal{D}[\delta x] \dots$$

So:

$$K(x_f, x_0, t) = \left( \int_{\delta x(0) = \delta x(t) = 0} \mathcal{D}[\delta x] e^{\frac{i}{\hbar} S[\delta x]} \right) e^{\frac{i}{\hbar} S[x_{cl}]}$$

$= A(t)$  indep. of  $x_0, x_f$

$$S[x_{cl}] = \frac{m\omega}{2 \sin \omega t} ((x_0^2 + x_f^2) \cos \omega t - 2x_0 x_f)$$



Determine  $A(t)$

shortcut (for  $0 < \omega t < \pi$ )

$$\begin{aligned} & \int dx K(x_f, x, t_2) K(x, x_0, t_1) \\ &= \int dx \langle x_f | e^{-\frac{i}{\hbar} \hat{H} t_2} | x \rangle \langle x | e^{-\frac{i}{\hbar} \hat{H} t_1} | x_0 \rangle \\ &= \langle x_f | e^{-\frac{i}{\hbar} \hat{H} (t_1 + t_2)} | x_0 \rangle \\ &= K(x_f, x_0, t_1 + t_2) \end{aligned}$$

Find  $A(t)$  such that

$$\int dx K(0, x, t_2) K(x, 0, t_1) = K(0, 0, t_1 + t_2)$$

Recall:

$$K(x_f, x_0, t) = A(t) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2 \sin \omega t} ((x_0^2 + x_f^2) \cos \omega t - 2x_0 x_f)\right)$$

$$\begin{aligned}
 \text{l.h.s.} &= \int dx A(t_2) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin\omega t_2} x^2 \cos\omega t_2\right) \\
 &\quad A(t_1) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin\omega t_1} x^2 \cos\omega t_1\right) \\
 &= A(t_1) A(t_2) \int dx \exp\left(\frac{i}{\hbar} \frac{m\omega}{2} \underbrace{\frac{\sin\omega t_1 \cos\omega t_2 + \sin\omega t_2 \cos\omega t_1}{\sin\omega t_1 \sin\omega t_2}}_{=\frac{\sin\omega(t_1+t_2)}{\sin\omega t_1 \sin\omega t_2}} x^2\right)
 \end{aligned}$$

$$\left[ \text{Recall Fresnel: } \int dx e^{iax^2} dx = \sqrt{\frac{\pi}{a}} e^{i\frac{\pi}{4}} \text{ (for } a > 0) \right]$$

$$= A(t_1) A(t_2) \sqrt{\frac{2\pi\hbar}{m\omega} \frac{\sin\omega t_1 \sin\omega t_2}{\sin\omega(t_1+t_2)}} e^{i\frac{\pi}{4}}$$

$$\stackrel{!}{=} A(t_1+t_2)$$

Rearrange:

$$A(t_1) \sqrt{\sin \omega t_1} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{i\frac{\pi}{4}} \cdot A(t_2) \sqrt{\sin \omega t_2} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{i\frac{\pi}{4}}$$
$$= A(t_1+t_2) \sqrt{\sin \omega (t_1+t_2)} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{i\frac{\pi}{4}}$$

simplest solution: all three terms = 1

i.e.

$$A(t) \sqrt{\sin \omega t} \sqrt{\frac{2\pi\hbar}{m\omega}} e^{i\frac{\pi}{4}} = 1$$

$$A(t) = \sqrt{\frac{m\omega}{2\pi\hbar \sin \omega t}} e^{-i\frac{\pi}{4}}$$

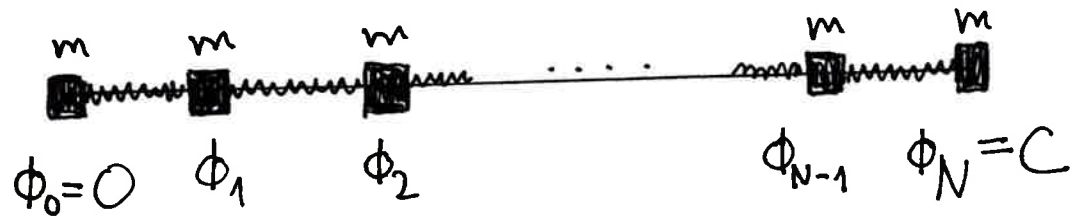
$$K(x_f, x_0, t) = \sqrt{\frac{m\omega}{2\pi\hbar \sin \omega t}} \exp\left(\frac{i}{\hbar} \frac{m\omega}{2 \sin \omega t} ((x_0^2 + x_f^2) \cos \omega t - 2x_0 x_f) - i\frac{\pi}{4}\right)$$

(for  $0 < \omega t < \pi$ )

but have to check!

## 2.4 Elastic chain

$N+1$  masses connected by springs:



Spring const:  $k$   
natural length:  $0$

$$\underline{\phi} = (\phi_1, \phi_2, \dots, \phi_{N-1})$$

Lagrangian:

$$L(\underline{\phi}, \dot{\underline{\phi}}) = \sum_{i=1}^{N-1} \frac{1}{2} m \dot{\phi}_i^2 - \sum_{i=0}^{N-1} \frac{1}{2} k (\phi_{i+1} - \phi_i)^2$$

Path integral:

$$K(\underline{\phi}_f, \underline{\phi}_0, t) = \langle \underline{\phi}_f | e^{-\frac{i}{\hbar} \hat{H} t} | \underline{\phi}_0 \rangle = \int \mathcal{D}[\underline{\phi}] e^{\frac{i}{\hbar} S[\underline{\phi}]}$$

$\underline{\phi}(t')$  with  
 $\underline{\phi}(0) = \underline{\phi}_0, \underline{\phi}(t) = \underline{\phi}_f$

$$S[\underline{\phi}] = \int_0^t dt' L(\underline{\phi}(t'), \dot{\underline{\phi}}(t')) dt'$$