Path integral

$$K(\underline{r}_f, \underline{r}_0, t) = \langle \underline{r}_f | e^{-\frac{i}{\hbar}\hat{H}t} | \underline{r}_0 \rangle = \int \underset{\substack{\uparrow \\ \text{int. over trajectories} \\ \text{from } \underline{r}_0 \text{ to } \underline{r}_f}}{\mathcal{D}}[\underline{r}]\, e^{\frac{i}{\hbar} S[\underline{r}]}$$

seen so far:

- for small times

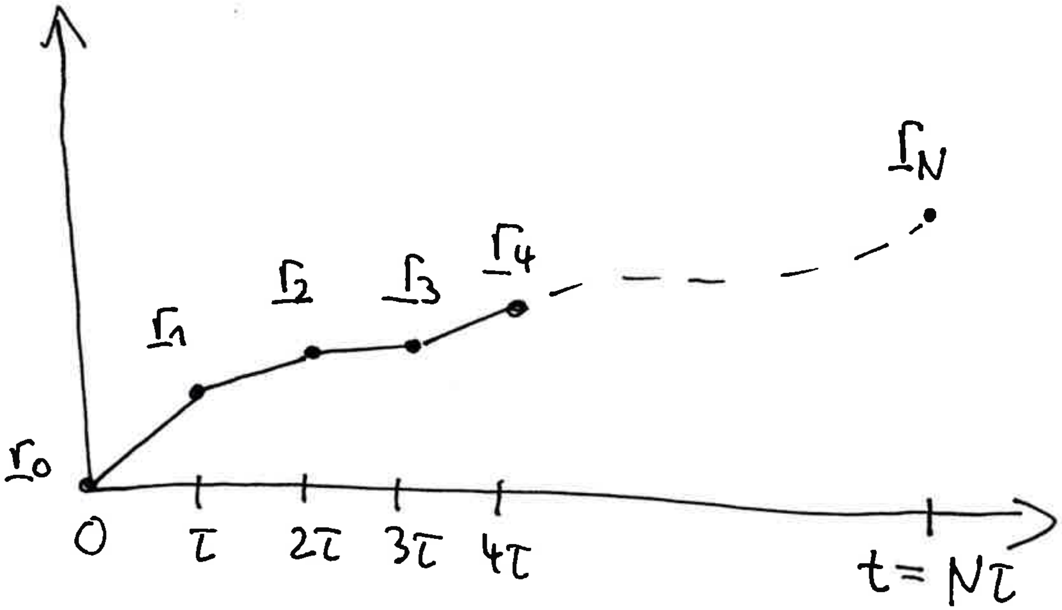
$$K(\underline{r}_f, \underline{r}_0, t) \simeq \left(\frac{m}{2\pi i \hbar t}\right)^{n/2} \exp\left(\frac{i}{\hbar}\left[\frac{1}{2}m\left(\frac{\underline{r}_f - \underline{r}_0}{t}\right)^2 - U(\underline{r}_0)\right] t\right)$$

- for arbitrary times

$$K(\underline{r}_f, \underline{r}_0, t) \simeq \left(\frac{mN}{2\pi i \hbar t}\right)^{nN/2} \int d^n r_1 \ldots \int d^n r_{N-1}\ \exp\left(\frac{i}{\hbar}\sum_{j=0}^{N-1}\left[\frac{1}{2}m\left(\frac{\underline{r}_{j+1} - \underline{r}_j}{\tau}\right)^2 - U(\underline{r}_j)\right]\tau\right)$$

$$\left(\begin{matrix} N \to \infty \\ \tau = \frac{t}{N} \end{matrix}\right)$$

interpret $\underline{r}_j$ as position after time $j\tau$



$$K(\underline{r}_f, \underline{r}_0, t) \simeq \left(\frac{mN}{2\pi i\hbar t}\right)^{nN/2} \int d^n r_1 \dots \int d^n r_{N-1} \, \exp\left(\frac{i}{\hbar} \sum_{j=0}^{N-1} \underbrace{\underbrace{\left[\frac{1}{2} m \left(\frac{\underline{r}_{j+1} - \underline{r}_j}{\tau}\right)^2 - U(\underline{r}_j)\right]}_{\substack{\text{good approx. for Lagrangian} \\ \text{in time interval from} \\ j\tau \text{ to } (j+1)\tau}} \tau}_{\text{good approx. for action}}\right)$$

$N \to \infty$

approximation becomes better and better

get integral over trajectories evaluated at all times

$$\sum_{j=0}^{N-1} \left[\frac{1}{2} m \left(\frac{\underline{r}_{j+1} - \underline{r}_j}{\tau}\right)^2 - U(\underline{r}_j)\right] \tau$$

$$\to \int_0^t L(\underline{r}(t'), \dot{\underline{r}}(t'))\, dt' = S[\underline{r}]$$

$$\boxed{K(\underline{r}_f, \underline{r}_0, t) = \int_{\substack{\underline{r}(0)=\underline{r}_0 \\ \underline{r}(t)=\underline{r}_f}} \mathcal{D}[\underline{r}]\; e^{\frac{i}{\hbar} S[\underline{r}]}}$$

with $\int \mathcal{D}[\underline{r}] \ldots = \lim_{N \to \infty} \left(\frac{m N}{2\pi i \hbar t}\right)^{nN/2} \int d^n r_1 \ldots \int d^n r_{N-1} \ldots$

Remarks

- rigorous treatment requires care with: integration measure, non-differentiable trajectories
- all trajectories needed
- action is stationary for trajectories satisfying eqs of motion

$$S[\underset{\substack{\uparrow \\ \text{satisfies} \\ \text{e.o.m}}}{r_{cl}} + \underset{\substack{\uparrow \\ \text{deviation}}}{\delta r}] = S[r_{cl}] + O((\delta r)^2)$$

traj. similar to classical ones give similar contribution

important joint contribution

$\Rightarrow$ classical trajectories dominate

- if some $\underline{r}$ are not possible, can have additional phases

- Hamiltonian mechanics version :

$$H = \underline{p} \cdot \dot{\underline{r}} - L \quad \Rightarrow \quad L = \underline{p} \cdot \dot{\underline{r}} - H$$

$$K(\underline{r}_f, \underline{r}_0, t) = \int \mathcal{D}[\underline{r}]\, \mathcal{D}[\underline{p}] \exp\left(\frac{i}{\hbar} \int_0^t \left(\underline{p}(t') \cdot \dot{\underline{r}}(t') - H(\underline{r}(t'), \underline{p}(t')\right) dt'\right)$$

trajectories in phase space
with $\underline{r}(0) = \underline{r}_0$, $\underline{r}(t) = \underline{r}_f$

2.3 Example: Harmonic Oscillator

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$S = \int_0^t \left(\frac{1}{2} m \dot{x}(t')^2 - \frac{1}{2} m \omega^2 x(t')^2\right) dt'$$

split $x(t')$ as:

$$x(t') = \underbrace{x_{cl}(t')}_{\substack{\text{solution of} \\ \text{classical laws} \\ \text{of motion}}} + \underbrace{\delta x(t')}_{\substack{\text{deviation} \\ \text{with} \\ \delta x(0) = \delta x(t) = 0}}$$

action:

$$S[x] = \int_0^t \left(\frac{1}{2} m \dot{x}_{cl}^2(t') - \frac{1}{2} m \omega^2 x_{cl}^2(t')\right) dt'$$

+ linear terms (must vanish due to stationarity of action)

$$+ \int_0^t \left(\frac{1}{2} m \,\delta\dot{x}(t')^2 - \frac{1}{2} m \omega^2 \,\delta x(t')^2\right) dt'$$

$$= S[x_{cl}] + S[\delta x]$$

determine x_{cl}:

$$\frac{\partial L}{\partial x} = \frac{d}{dt'} \frac{\partial L}{\partial \dot{x}}$$

$$-m\omega^2 x_{cl} = m\ddot{x}_{cl}$$

$$x_{cl}(t') = A \sin \omega t' + B \cos \omega t'$$

$$S[x_{cl}] = \frac{m\omega}{2} \left[(A^2 - B^2) \frac{\sin 2\omega t}{2} + 2AB \frac{\cos 2\omega t - 1}{2} \right]$$

boundary conditions:

$$x_{cl}(0) = B \stackrel{!}{=} x_0$$

$$x_{cl}(t) = A \sin \omega t + B \cos \omega t \stackrel{!}{=} x_f \Rightarrow A = \frac{x_f - B \cos \omega t}{\sin \omega t}$$

$$S[x_{cl}] = \frac{m\omega}{2 \sin \omega t} \left((x_0^2 + x_f^2) \cos \omega t - 2 x_0 x_f \right)$$

Result so far

$$K(x_f, x_0, t) = \int \mathcal{D}[x]\, e^{\frac{i}{\hbar}(S[x_{cl}] + S[\delta x])}$$

would like $\uparrow$ $\mathcal{D}[\delta x]$

$$\int \mathcal{D}[x] \ldots = \lim_{N\to\infty} \left(\frac{mN}{2\pi i\hbar t}\right)^{\frac{N}{2}} \int dx_1 \ldots dx_{N-1} \ldots$$

$$\underset{x_j = x_{cl,j} + \delta x_j}{=} \lim_{N\to\infty} \left(\frac{mN}{2\pi i\hbar t}\right)^{\frac{N}{2}} \int d\delta x_1 \ldots d\delta x_{N-1} \ldots$$

$$= \int \mathcal{D}[\delta x] \ldots$$

So:

$$K(x_f, x_0, t) = \underbrace{\left(\int_{\delta x(0) = \delta x(t) = 0} \mathcal{D}[\delta x]\, e^{\frac{i}{\hbar} S[\delta x]} \right)}_{= A(t) \text{ indep. of } x_0, x_f} e^{\frac{i}{\hbar} S[x_{cl}]}$$

$$S[x_{cl}] = \frac{m\omega}{2\sin\omega t}\left((x_0^2 + x_f^2)\cos\omega t - 2x_0 x_f\right)$$

Determine $A(t)$

shortcut (for $0<\omega t<\pi$)

$$\int dx\, K(x_f, x, t_2)\, K(x, x_0, t_1)$$

$$= \int dx \langle x_f | e^{-\frac{i}{\hbar}\hat{H}t_2} | x\rangle \langle x | e^{-\frac{i}{\hbar}\hat{H}t_1} | x_0\rangle$$

$$= \langle x_f | e^{-\frac{i}{\hbar}\hat{H}(t_1+t_2)} | x_0\rangle$$

$$= K(x_f, x_0, t_1+t_2)$$

Find $A(t)$ such that

$$\boxed{\int dx\, K(0, x, t_2)\, K(x, 0, t_1) = K(0, 0, t_1+t_2)}$$

Recall:

$$\boxed{K(x_f, x_0, t) = A(t) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin\omega t}\left((x_0^2 + x_f^2)\cos\omega t - 2x_0 x_f\right)\right)}$$

$$\text{l.h.s.} = \int dx \; A(t_2) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin\omega t_2} x^2 \cos\omega t_2\right)$$

$$A(t_1) \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin\omega t_1} x^2 \cos\omega t_1\right)$$

$$= A(t_1) A(t_2) \int dx \; \exp\left(\frac{i}{\hbar} \frac{m\omega}{2} \underbrace{\frac{\sin\omega t_1 \cos\omega t_2 + \sin\omega t_2 \cos\omega t_1}{\sin\omega t_1 \; \sin\omega t_2}}_{= \frac{\sin\omega(t_1+t_2)}{\sin\omega t_1 \; \sin\omega t_2}} x^2\right)$$

[Recall Fresnel: $\int dx \; e^{iax^2} dx = \sqrt{\frac{\pi}{a}} \, e^{i\frac{\pi}{4}}$ (for $a>0$)]

$$= A(t_1) A(t_2) \sqrt{\frac{2\pi\hbar}{m\omega} \frac{\sin\omega t_1 \; \sin\omega t_2}{\sin\omega(t_1+t_2)}} \; e^{i\frac{\pi}{4}}$$

$$\overset{!}{=} A(t_1+t_2)$$

Rearrange:

$$A(t_1)\sqrt{\sin\omega t_1}\sqrt{\frac{2\pi\hbar}{m\omega}}\,e^{i\frac{\pi}{4}}\cdot A(t_2)\sqrt{\sin\omega t_2}\sqrt{\frac{2\pi\hbar}{m\omega}}\,e^{i\frac{\pi}{4}}$$

$$= A(t_1+t_2)\sqrt{\sin\omega(t_1+t_2)}\sqrt{\frac{2\pi\hbar}{m\omega}}\,e^{i\frac{\pi}{4}}$$

simplest solution: all three terms = 1

i.e.

$$A(t)\sqrt{\sin\omega t}\sqrt{\frac{2\pi\hbar}{m\omega}}\,e^{i\frac{\pi}{4}} = 1$$

$$A(t) = \sqrt{\frac{m\omega}{2\pi\hbar\sin\omega t}}\,e^{-i\frac{\pi}{4}}$$

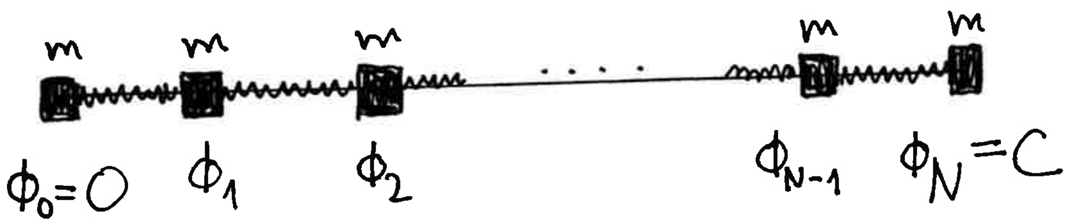
$$\boxed{K(x_f, x_0, t) = \sqrt{\frac{m\omega}{2\pi\hbar\sin\omega t}}\exp\left(\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left((x_0^2+x_f^2)\cos\omega t - 2x_0 x_f\right) - i\frac{\pi}{4}\right)}$$

(for $0 < \omega t < \pi$)

but have to check!

2.4 Elastic chain

$N+1$ masses connected by springs:



spring const: k
natural length: 0

$$\underline{\Phi} = (\phi_1, \phi_2, \dots, \phi_{N-1})$$

Lagrangian:

$$L(\underline{\Phi}, \dot{\underline{\Phi}}) = \sum_{i=1}^{N-1} \frac{1}{2} m \dot{\phi}_i^2 - \sum_{i=0}^{N-1} \frac{1}{2} k (\phi_{i+1} - \phi_i)^2$$

Path integral:

$$K(\underline{\Phi}_f, \underline{\Phi}_0, t) = \langle \underline{\Phi}_f | e^{-\frac{i}{\hbar}\hat{H}t} | \underline{\Phi}_0 \rangle = \int D[\underline{\Phi}]\, e^{\frac{i}{\hbar} S[\underline{\Phi}]}$$

$\underline{\Phi}(t')$ with
$\underline{\Phi}(0) = \underline{\Phi}_0$, $\underline{\Phi}(t) = \underline{\Phi}_f$

$$S[\underline{\Phi}] = \int_0^t dt'\, L(\underline{\Phi}(t'), \dot{\underline{\Phi}}(t'))\, dt'$$