# Two indistinguishable particles

## wavefunctions

$\psi(\underline{r}_1, \underline{r}_2)$ with $\psi(\underline{r}_2, \underline{r}_1) = \pm\, \psi(\underline{r}_1, \underline{r}_2)$

← bosons

↖ fermions

## basis

for single particle $\quad \psi_i(\underline{r})$

$|i\rangle$

for two particles $\quad \psi_{ij}(\underline{r}_1, \underline{r}_2) = C\left(\psi_i(\underline{r}_1)\psi_j(\underline{r}_2) \pm \psi_j(\underline{r}_1)\psi_i(\underline{r}_2)\right)$

$|i, j\rangle = C\left(|i\rangle|j\rangle \pm |j\rangle|i\rangle\right)$

$C = \frac{1}{\sqrt{2}}$ if $i \neq j$

## 4.2 N particles

**wavefunctions** $\psi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$

for bosons: remains the same if two args. are interch.

for fermions: flips sign " " " " "

**basis**

$$\underbrace{\psi_{i_1 i_2 \ldots i_N}}_{\text{occupied single-particle states}} (\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N) = C\big(\psi_{i_1}(\underline{r}_1)\, \psi_{i_2}(\underline{r}_2) \ldots \psi_{i_N}(\underline{r}_N)$$

+ terms where indices/args. are interchanged )

↑

for fermions each exchange of two indices/ arguments leads to a minus sign

# permutations

bijective mappings between n integers e.g. $1 \to 2, 2 \to 1, 3 \to 4, 4 \to 3$

every permutation can be written as successive application of **transpositions** (permutations exchanging two elements)

$$\operatorname{sgn} \pi = \begin{cases} 1 & \text{for even number of transpositions} \\ -1 & \text{" odd " " "} \end{cases}$$

$$\psi_{i_1 i_2 \dots}(\underline{r}_1, \underline{r}_2, \dots) = C \sum_{\pi} \psi_{i_{\pi(1)}}(\underline{r}_1)\, \psi_{i_{\pi(2)}}(\underline{r}_2) \dots \times \begin{cases} 1 & \text{for bosons} \\ \operatorname{sgn}\pi & \text{for fermions} \end{cases}$$

$$|i_1, i_2, \dots\rangle = C \sum_{\pi} |i_{\pi(1)}\rangle |i_{\pi(2)}\rangle \dots \times \begin{cases} 1 & \text{for bosons} \\ \operatorname{sgn}\pi & \text{for fermions} \end{cases}$$

## normalisation

for fermions:

$$C = \frac{1}{\sqrt{N!}}$$

for bosons

$$C = \frac{1}{\sqrt{N! \prod_i n_i!}}$$

where $n_i$ = number of particles in single-particle state $i$

# 4.3 Fock space

can have states that are linear combinations of states with different number of particles

$\mathcal{F}_N$ = Hilbert space of states with N particles

overall Hilbert space contains lin. comb. of states from different $\mathcal{F}_N$

$$\bigoplus_{N=0}^{\infty} \mathcal{F}_N = \text{set of lin. comb. of elements of all } \mathcal{F}_N$$
$$= \text{Fock space}$$

Contains vacuum state = state with no particles

## 4.4 Creation + annihilation operators (for bosons)

occupation number representation

$$|\underbrace{1,\dots,1}_{n_1},\underbrace{2,\dots,2}_{n_2},\dots\rangle = |n_1, n_2, \dots\rangle$$

corresponding bra: $\langle n_1, n_2, \dots|$

$$\boxed{\text{Creation operator:} \quad a_i^\dagger |\dots n_i \dots\rangle = \sqrt{n_i+1}\, |\dots n_i+1 \dots\rangle}$$

$$\boxed{\text{annihilation operator:} \quad a_i |\dots n_i \dots\rangle = \sqrt{n_i}\, |\dots n_i-1 \dots\rangle}$$

Square root factors

- guarantee that $a_i |\dots 0 \dots\rangle = 0$
- consistency with raising/lowering operators for harm. oscillator
- $a_i, a_i^\dagger$ mutually adjoint

# Commutators $[A,B] = AB - BA$

**Thm:** $[a_i, a_j] = 0$, $[a_i^\dagger, a_j^\dagger] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$

**Proof:**

- operators with different indices change different occupation numbers and thus commute
- $[a_i, a_i] = 0$, $[a_i^\dagger, a_i^\dagger] = 0$
- $[a_i, a_i^\dagger] |\ldots n_i \ldots\rangle$

$$= a_i a_i^\dagger \, |\ldots n_i \ldots\rangle - a_i^\dagger a_i |\ldots n_i \ldots\rangle$$

$$= a_i \sqrt{n_i+1} \, |\ldots n_i+1 \ldots\rangle - a_i^\dagger \sqrt{n_i} \, |\ldots n_i-1 \ldots\rangle$$

$$= (n_i + 1) \, |\ldots n_i \ldots\rangle - n_i \, |\ldots n_i \ldots\rangle$$

$$= 1 \, |\ldots n_i \ldots\rangle$$

# Occupation number operator

$$a_i^\dagger a_i \,|\ldots n_i \ldots\rangle = n_i \,|\ldots n_i \ldots\rangle$$

# 4.5 Hamiltonian in 2nd quantisation

**Example:** system with discrete sites

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 \end{matrix}$$

## Single particle Hamiltonian

wave function $\begin{pmatrix} \psi(1) \\ \psi(2) \\ \vdots \end{pmatrix}$

normalisation

$$\sum_i |\psi(i)|^2 = 1$$

Continuous case:

$$(\hat{H}\psi)(x) = \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x) \right) \psi(x)$$

here:

$$\psi'(x) \to \psi(x+1) - \psi(x)$$

$$\psi''(x) \to \psi'(x) - \psi'(x-1)$$

$$= (\psi(x+1) - \psi(x)) - (\psi(x) - \psi(x-1))$$

$$= \psi(x+1) - 2\psi(x) + \psi(x-1)$$

Hamiltonian:

$$(\hat{H}\psi)(x) = -\frac{\hbar^2}{2m}\left(\psi(x+1) - 2\psi(x) + \psi(x-1)\right) + U(x)\psi(x)$$

in matrix form:

$$\hat{H} = -\frac{\hbar^2}{2m}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} U(1) & & \\ & U(2) & \\ & & \ddots \end{pmatrix}$$

# Multiple particle Hamiltonian

wavefunctions $\psi(x_1, x_2, \dots)$

continuous case:

e.g. $$\hat{H}_{mult} = \frac{\hat{\underline{p}}_1^2}{2m} + U(\underline{r}_1) + \frac{\hat{\underline{p}}_2^2}{2m} + U(\underline{r}_2) + U^{int}(\underline{r}_1, \underline{r}_2)$$

discrete version:

$$\hat{H}_{mult} = \sum_p \underbrace{\hat{H}_p}_{\substack{\uparrow \\ \text{acts} \\ \text{w.r.t. } x_p}} + \frac{1}{2} \sum_{\substack{p=1..N \\ p'=1..N \\ p \neq p'}} U^{int}(x_p, x_{p'})$$

# now with creation + annihilation operators

basis: consider states localised at sites

$$\varphi_i(x) = \delta_{xi} \qquad \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \dots$$

## • potential

$$\sum_i \underset{\substack{\uparrow \\ \text{diag. el.}}}{U(i)} \; \underbrace{a_i^+ \, a_i}_{\text{occupation number}}$$

## • full single particle Hamiltonian

$$\sum_{ij} H(i,j) \, a_i^+ \, a_j$$

(no proof)

- interaction potential

$$\frac{1}{2}\sum_{ij} U^{int}(i,j)\underbrace{a_i^+ a_i}_{\substack{\text{particles}\\ \text{at } i}}\,(\underbrace{a_j^+ a_j}_{\substack{\text{particles}\\ \text{at } j}} - \underbrace{\delta_{ij}}_{\substack{\text{no self}\\ \text{interaction}}})$$

Thm: $a_i^+ a_i (a_j^+ a_j - \delta_{ij}) = a_i^+ a_j^+ a_j a_i$

Proof: $a_i^+ a_j^+ a_j a_i$

$$= a_i^+ a_j^+ a_i a_j$$

$$= a_i^+ [a_i a_j^+ - \underbrace{(a_i a_j^+ - a_j^+ a_i)}_{=[a_i, a_j^+] = \delta_{ij}}] a_j$$

$$= a_i^+ a_i (a_j^+ a_j - \delta_{ij}) \qquad \square$$

Total:

$$\hat{H}_{mult} = \sum_{ij} H(i,j)\, a_i^\dagger a_j + \frac{1}{2} \sum_{ij} U^{int}(i,j)\, a_i^\dagger a_j^\dagger a_j a_i$$

# Continuous case

$i \to \underline{r}, \quad j \to \underline{r}'$

$a_i^\dagger \to a^\dagger(\underline{r}), \quad a_i \to a(\underline{r}), \quad [a(\underline{r}), a^\dagger(\underline{r}')] = \delta(\underline{r}-\underline{r}')$

## Hamiltonian

$$\hat{H} = \int d^n r \; a^\dagger(\underline{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + \mathcal{U}(\underline{r})\right) a(\underline{r})$$

$$+ \frac{1}{2}\int d^n r \int d^n r' \; \mathcal{U}^{int}(\underline{r},\underline{r}') \, a^\dagger(\underline{r}) a^\dagger(\underline{r}') a(\underline{r}') \, a(\underline{r})$$

Single particle term looks similar to exp. value

$$\int d^n r \;\; \psi^*(\underline{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + \mathcal{U}(\underline{r})\right)\psi(\underline{r})$$

as if we replaced wavef. by operator (second quantisation)