Path integral in 2nd quantisation

(for bosons)

$$\operatorname{tr} e^{-\frac{i}{\hbar}\hat{H}t} = \int\limits_{a_j(t)=a_j(0)} \mathcal{D}[a_1,a_2,\ldots] \exp\Big(-\int_0^t dt' \sum_j a_j^*(t')\,\dot{a}_j(t')$$
$$-\frac{i}{\hbar}\int_0^t dt'\, H(a_1(t'),a_2(t'),\ldots,a_1^*(t'),a_2^*(t'),\ldots)\Big)$$

5.2 Fermions

Creation & annihilation operators anticommute

(apart from $[a_j, a_j^\dagger]_+ = 1$)

path integral should involve anticommuting "numbers"

deal with complex conjugation later

Def: Grassmannian variables $\eta_1 \dots \eta_N$ satisfy

$$\eta_i \eta_j + \eta_j \eta_i = 0$$

i.e. $\eta_i \eta_j = - \eta_j \eta_i$

Corollary $\eta_i\eta_i + \eta_i\eta_i = 0 \Rightarrow \eta_i^2 = 0$

possible functions

$$f(\eta_i) = a + b\,\eta_i$$

$$g(\eta_i, \eta_j) = a + b\,\eta_i + c\,\eta_j + d\,\eta_i\eta_j$$

Def: Derivative

- $\frac{\partial}{\partial \eta_i}\eta_i = 1$
- $\frac{\partial}{\partial \eta_i}$ acting on term indep. of $\eta_i = 0$
- $\frac{\partial}{\partial \eta_i}$ anticommutes (needed as $\frac{\partial}{\partial \eta_i}(\eta_j\eta_i) = -\frac{\partial}{\partial \eta_i}\eta_i\eta_j = -\eta_j$)

Def: Integral

integral = derivative, i.e.

- $\int d\eta_i \, \eta_i = 1$
- $\int d\eta_i$ acting on term indep. of $\eta_i = 0$
- $\int d\eta_i$ anticommutes with Grassmannians / integrals / derivatives

Products of 2 Grassmannians commute

$$(\eta_i \eta_j)\, \eta_k = - \eta_i \eta_k \eta_j = \eta_k (\eta_i \eta_j)$$

Complex conjugation

Regard η_i and η_i^* as independent Grassmannians

Gauss integral (1D)

$$\int d\eta^* \int d\eta \; e^{-a\eta^*\eta}$$

$$= \int d\eta^* \int d\eta \left(1 - a\eta^*\eta + \frac{1}{2}(a\eta^*\eta)^2 - \dots\right)$$

$$= a \int d\eta^* \int d\eta \; \eta\, \eta^*$$

$$= a$$

complex case:

$$\underbrace{\int_{-\infty}^{\infty} \frac{d\mathrm{Re}\,z \; d\mathrm{Im}\,z}{\pi}}_{= \int dz^* \int dz} e^{-a|z|^2} = \frac{1}{\pi}\sqrt{\frac{\pi}{a}}\sqrt{\frac{\pi}{a}} = \frac{1}{a}$$

Gauss integral (multi-dimensional)

Complex case: $\int dz_1^* dz_1 \dots dz_N^* dz_N \; e^{-\underline{z}^+ A \underline{z}} = \dfrac{1}{\det A}$

Grassmannians: $\boxed{I = \int d\eta_1^* d\eta_1 \dots d\eta_N^* d\eta_N \; e^{-\eta^+ A \eta} = \det A}$

$$I = \int d\eta_1^* d\eta_1 \dots d\eta_N^* d\eta_N \; \exp\Big(-\sum_{ij} \eta_i^* A_{ij} \eta_j\Big)$$

$$= \int d\eta_1^* d\eta_1 \dots d\eta_N^* d\eta_N \; \frac{1}{N!} \Big(-\sum_{ij} \eta_i^* A_{ij} \eta_j\Big)^N$$

only terms with one η_1, η_1^* etc contribute

$$= \int d\eta_1 \, d\eta_1^* \dots d\eta_N \, d\eta_N^* \; \frac{1}{N!} \Big(\sum_{ij} \eta_i^* A_{ij} \eta_j\Big)^N$$

differentials exchanged

order terms in increasing order of $i \Rightarrow$ factor $N!$

as each i and j appears once: $j = \underset{\substack{\uparrow \\ \text{permutation}}}{\pi}(i)$

$$I = \int d\eta_1 d\eta_1^* \dots d\eta_N d\eta_N^* \sum_{\pi} \prod_{i=1}^{N} \eta_i^* A_{i\pi(i)} \eta_{\pi(i)}$$

reorder η's using $\eta_1^* \eta_2 \eta_2^* \eta_1$

$$= \eta_1^* \eta_1 \eta_2 \eta_2^*$$

$$= - \eta_1^* \eta_1 \eta_2^* \eta_2$$

$$I = \int d\eta_1 d\eta_1^* \dots d\eta_N d\eta_N^* \sum_{\pi} \operatorname{sgn} \pi \prod_{i=1}^{N} \eta_i^* A_{i\pi(i)} \eta_i$$

evaluate integrals using $\int d\eta_i \, d\eta_i^* \; \eta_i^* \eta_i = 1$:

$$I = \sum_{\pi} \operatorname{sgn} \pi \prod_{i=1}^{N} A_{i\,\pi(i)}$$

$$= \det A \qquad \square$$

Perturbation theory

average: $\langle \dots \rangle = \frac{1}{c} \int d\eta_1^* \, d\eta_1 \dots d\eta_N^* \, d\eta_N \, e^{-\underline{\eta}^\dagger A \underline{\eta}} \dots$

$(c = \det A)$

1d case $(A=a)$:

$$\langle \eta \eta^* \rangle = \frac{1}{a} \int d\eta^* \, d\eta \; \eta \eta^* \; e^{-a \eta^* \eta}$$

$$= \frac{1}{a}$$

$$\langle \eta \eta \rangle = 0$$

$$\langle \eta^* \eta^* \rangle = 0$$

Wick thm: To compute average of product of Grassmannians

- sum over ways to contract η's and η^*'s
- reorder to place contracted factors next to each other as $\eta_j \, \eta^*_k$
- $\eta_j \, \eta^*_k$ gives $(A^{-1})_{jk}$

Example: $\langle \eta_{j_1} \eta^*_{j_2} \eta^*_{j_3} \eta_{j_4} \rangle = \langle \eta_{j_1} \eta^*_{j_2} \eta^*_{j_3} \eta_{j_4} \rangle + \langle \eta_{j_1} \eta^*_{j_2} \eta^*_{j_3} \eta_{j_4} \rangle$

$$= - \langle \eta_{j_1} \eta^*_{j_2} \eta_{j_4} \eta^*_{j_3} \rangle + \langle \eta_{j_1} \eta^*_{j_3} \eta_{j_4} \eta^*_{j_2} \rangle$$

$$= -(A^{-1})_{j_1 j_2} (A^{-1})_{j_4 j_3} + (A^{-1})_{j_1 j_3} (A^{-1})_{j_4 j_2}$$

Path integral for fermionic systems

$$\boxed{\begin{aligned}\mathrm{tr}\, e^{-\frac{i}{\hbar}\hat{H}t} = \int D[a_1, a_2, \ldots] \exp\Big(&-\int_0^t dt' \sum_j a_j^*(t')\, \dot{a}_j(t') \\ &-\frac{i}{\hbar}\int_0^t dt'\, H(a_1(t'), a_2(t'), \ldots, a_1^*(t'), a_2^*(t'), \ldots)\Big)\end{aligned}}$$

integral over Grassmannians $a_j(t'),\ a_j^*(t')$

boundary cond: $a_j(t) = -a_j(0),\quad a_j^*(t) = -a_j^*(0)$

integration measure:

$$\int D[a_1, a_2, \ldots] = \lim_{N\to\infty} \prod_j \prod_{k=0}^{N-1} \int da_{jk}^*\, da_{jk} \cdots$$

time steps

e.g. site index

time steps

independent Grassmann variables

Example: fermionic harmonic oscillator

$$\hat{H} = \hbar \omega a^\dagger a$$

(ground state energy can be added)

$$I = \mathrm{tr}\, e^{-\frac{i}{\hbar}\hat{H}t} = \int \mathcal{D}[a] \exp\left(\int_0^t dt' \left(-a^*(t')\dot{a}(t') - \frac{i}{\hbar}\hbar\omega a^*(t')a(t')\right)\right)$$

discrete version:

$$I = \lim_{N\to\infty} \int da_0^* da_0 \dots da_{N-1}^* da_{N-1} \exp\left(-\sum_{k=0}^{N-1} a_{k+1}^* \frac{a_{k+1}-a_k}{\tau}\tau - i\omega \sum_{k=0}^{N-1} a_{k+1}^* a_k \tau\right)$$

where $\tau = \frac{t}{N}$, $a_N = -a_0$, $a_N^* = -a_0^*$

$$I = \lim_{N\to\infty} \int da_0^* da_0 \dots da_{N-1}^* da_{N-1} \exp\left(-\sum_{k=0}^{N-1} a_{k+1}^* a_{k+1} + (1-i\omega\tau)\sum_{k=0}^{N-1} a_{k+1}^* a_k\right)$$

$$I = \lim_{N\to\infty} \int da_0^* da_0 \dots da_{N-1}^* da_{N-1} \exp\left(\sum_{k=0}^{N-1} a_k a_k^* + (1-i\omega\tau)\sum_{k=0}^{N-1} a_{k+1}^* a_k\right)$$

$$= \lim_{N\to\infty} \int da_0^* da_0 \dots da_{N-1}^* da_{N-1} \prod_{k=0}^{N-1}(1+a_k a_k^*) \prod_{k=0}^{N-1}\left(1+(1-i\omega\tau)a_{k+1}^* a_k\right)$$

each Grassmannian must appear linearly,
this happens if we combine all linear terms from
first or second factor

$$I = I_1 + I_2$$

$$I_1 = \lim_{N\to\infty} \int da_0^* da_0 \dots da_{N-1}^* da_{N-1} \prod_{k=0}^{N-1} a_k a_k^* = 1$$

$$I_2 = \lim_{N\to\infty} \int da_0^* \, da_0 \dots da_{N-1}^* \, da_{N-1} \prod_{k=0}^{N-1} (1-i\omega\tau)\, a_{k+1}^* a_k$$

moved da_0^* here
compensate sign factor by writing da_N^*

$$= \lim_{N\to\infty} \int (da_0 \, da_1^*) \dots (da_{N-1} \, da_N^*) \prod_{k=0}^{N-1} (1-i\omega\tau)\, a_{k+1}^* a_k$$

$$= \lim_{N\to\infty} (1-i\omega\tau)^N$$

$$= \lim_{N\to\infty} \left(1-\frac{i\omega t}{N}\right)^N = e^{-i\omega t}$$

$$\mathrm{tr}\, e^{-\frac{i}{\hbar}\hat{H}t} = 1 + e^{-i\omega t}$$

direct calculation:

$$\hat{H} = \hbar\omega a^{+} a$$

$$\hbar\omega a^{+} a|0\rangle = 0$$

$$\hbar\omega a^{+} a|1\rangle = \hbar\omega|1\rangle$$

$$\mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}E_1 t} + e^{-\frac{i}{\hbar}E_2 t} = 1 + e^{-i\omega t}$$