Problem class I: Recall the Hamiltonian mechanics path integral.

$$K(\underline{r}_f, \underline{r}_0, t) = \int_{\substack{\underline{r}(0)=\underline{r}_0 \\ \underline{r}(t)=\underline{r}_f}} \mathcal{D}[\underline{r}]\,\mathcal{D}[\underline{p}]\; e^{\frac{i}{\hbar}S[\underline{r},\underline{p}]}$$

$$S[\underline{r},\underline{p}] = \int (\underline{p}(t')\cdot\dot{\underline{r}}(t') - H(\underline{r}(t'),\underline{p}(t')))\,dt'$$

Q: Find the propagater of the free particle with Hamiltonian

$$H = \frac{p^2}{2m}$$

in this formalism

A: We start by splitting

$$x(t') = x_{cl}(t') + \delta x(t')$$

$$p(t') = p_{cl}(t') + \delta p(t')$$

$$S[x,p] = \int_0^t (p(t')\cdot\dot{x}(t') - \frac{p(t')^2}{2m})\,dt'$$

$$= \int_0^t (p_{cl}(t')\,\dot{x}_{cl}(t') - \frac{p_{cl}(t')^2}{2m})\,dt'$$

+ linear terms (vanish due to stationarity of action)

$$+ \int_0^t (\delta p(t')\,\delta\dot{x}(t') - \frac{\delta p(t')^2}{2m})\,dt'$$

$$= S[x_{cl}, p_{cl}] + S[\delta x, \delta p]$$

class. solution:

$$p(t') = m\,\frac{x_f - x_0}{t} = \text{const}$$

$$x(t') = x_0 + \frac{x_f - x_0}{t}\,t'$$

(check units

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = 0 \Rightarrow p = \text{const}$$
)

$$S[x_{cl}, p_{cl}] = \int \left(m\left(\frac{x_f - x_0}{t}\right)^2 - \frac{1}{2m}\,m^2\left(\frac{x_f - x_0}{t}\right)^2\right)dt'$$

$$= \frac{1}{2}\,m\left(\frac{x_f - x_0}{t}\right)^2 t$$

$$K(x_f, x_0, t) = \int \mathcal{D}[x]\mathcal{D}[p] \; e^{\frac{i}{\hbar}\left(S[x_{cl}, p_{cl}] + S[\delta x, \delta p]\right)}$$

$$= \underbrace{\Big(\int \underset{\substack{\uparrow \\ \text{Jacobian}=1}}{\mathcal{D}}[\delta x]\mathcal{D}[\delta p] \; e^{\frac{i}{\hbar}S[\delta x, \delta p]} \Big)}_{= A(t)} e^{\frac{i}{\hbar}S[x_{cl}, p_{cl}]}$$

$\delta x(0) = \delta x(t) = 0$

Evaluate $A(t)$

to be determined in Q2a

$$A(t) = \lim_{N\to\infty} C_N \int d\delta x_1 \dots d\delta x_{N-1} \, d\delta p_0 \dots d\delta p_{N-1}$$

$$\exp\Big(\frac{i}{\hbar} \underbrace{\sum_{j=0}^{N-1} \Big(\delta p_j \frac{\delta x_{j+1} - \delta x_j}{\tau} - \frac{\delta p_j^2}{2m} \Big)\tau}_{\sum_{j=1}^{N-1} \delta p_{j-1}\delta x_j - \sum_{j=\cancel{0}1}^{N-1}\delta p_j \delta x_j - \sum_{j=0}^{N-1}\frac{\delta p_j^2}{2m}\tau} \Big)$$

(*)

$$= \lim_{N\to\infty} C_N \int d\delta p_0 \dots d\delta p_{N-1}$$

$$\prod_{j=1}^{N-1} \underbrace{\int d\delta x_j \exp\Big(\frac{i}{\hbar}(\delta p_{j-1} - \delta p_j)\delta x_j\Big)}_{2\pi\hbar\,\delta(\delta p_{j-1} - \delta p_j)} \exp\Big(-\frac{i}{\hbar}\sum_{j=0}^{N-1}\frac{\delta p_j^2}{2m}\tau\Big)$$

$$\left(\frac{1}{2\pi\hbar}\int e^{ipx/\hbar} dp = \delta(x) \right)$$

$$= \lim_{N\to\infty} C_N (2\pi\hbar)^{N-1} \underbrace{\int d\delta p_0 \exp\Big(-\frac{i}{\hbar}\frac{\delta p_0^2}{2m}\underbrace{N\tau}_{=t}\Big)}_{\sqrt{\frac{\pi}{i\frac{1}{\hbar}\frac{1}{2m}t}}}$$

$$\left(\int e^{-iax^2} dx = \sqrt{\frac{\pi}{ia}} \right)$$

$$= \lim_{N\to\infty} C_N (2\pi\hbar)^{N-1} \sqrt{\frac{2\pi\hbar m}{it}}$$

So

$$K(x_f, x_0, t) = \lim_{N\to\infty} C_N \, (2\pi\hbar)^{N-1} \sqrt{\frac{2\pi\hbar m}{it}} \exp\left(\frac{i}{\hbar}\frac{1}{2}m\frac{(x_f-x_0)^2}{t}\right)$$

If we had not split off the classical solution the path integral for $K(x_f, x_0, t)$ could be evaluated in a way similar to the integral for $A(t)$. But we'd have to use x, p instead of $\delta x, \delta p$ and we would have to keep the terms crossed out in (*) as x_0 and x_N may be different from zero whereas $\delta x_0 = \delta x_N = 0$. This would lead to an extra linear term and hence a shifted Fresnel integral. The final results for both approaches coincide.