AQT problem class 3

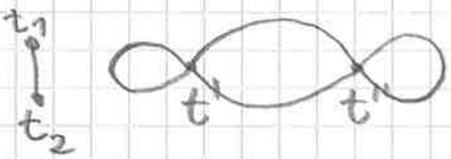
$$\langle x(t_1)\, x(t_2)\, \exp\left(-\frac{i\varepsilon}{\hbar}\int_0^t dt'\, x(t')^3\right)\rangle$$
$$= \langle x(t_1)\, x(t_2)\left(1 + \frac{1}{2}\left(-\frac{i\varepsilon}{\hbar}\right)^2 \int_0^t dt' \int_0^t dt''\, x(t')^3\, x(t'')^3 + O(\varepsilon^4)\right)\rangle$$

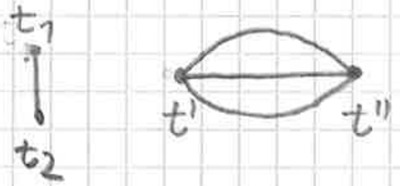
as averages of products with an odd number of factors vanish

$$\langle x(t_1)\, x(t_2)\rangle = iG(t_1, t_2)$$

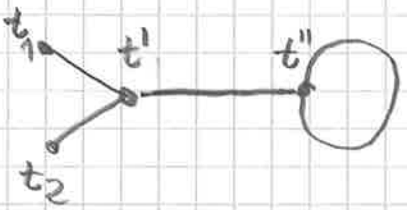
Diagrams contributing to $\int_0^t dt' \int_0^t dt'' \langle x(t_1)\, x(t_2)\, x(t')^3\, x(t'')^3\rangle$

A) $x(t_1)$ contracted with $x(t_2)$

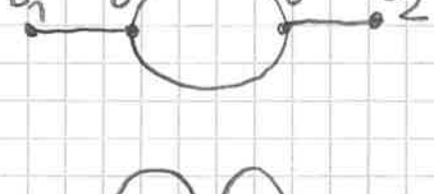
9 

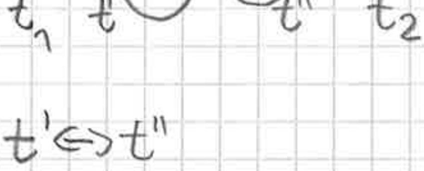
6 

B) $x(t_1)$ and $x(t_2)$ both contracted with $x(t')$ (or both contracted with $x(t'')$)

$18 = 3\cdot 2\cdot 3$  $+\ (t' \leftrightarrow t'')$

C) $x(t_1)$ contracted with $x(t')$, $x(t_2)$ contracted with $x(t'')$ (or the other way around)

$18 = 3\cdot 3\cdot 2$ 

$9 = 3\cdot 3$ 

and $t' \leftrightarrow t''$

check: altogether $15 + 36 + 54 = 105$ diagrams

$G(t',t'')$ has to satisfy

$$-\frac{m}{\hbar}\left(\frac{\partial^2}{\partial t'^2}+\omega^2\right)G(t',t'')=\delta(t'-t'')$$

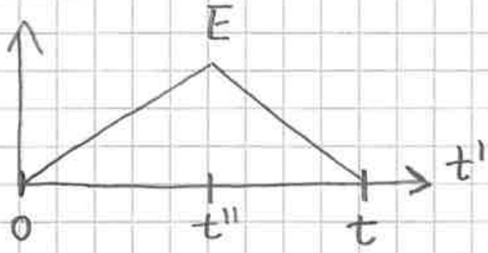
$$G(0,t'')=G(t,t'')=0 \qquad (*)$$

Solve this for $\omega=0$

For $t'\neq t''$ we have

$$\frac{\partial^2}{\partial t'^2}G(t',t'')=0 \quad\Rightarrow\quad G(t',t'')=A+Bt'$$

where the coefficients may depends on whether $t'\leq t''$ or $t'>t''$. Demanding continuity as well as $(*)$ we obtain



$$G(t',t'')=\begin{cases}E\frac{t'}{t''} & t'\leq t''\\ E\frac{t-t'}{t-t''} & t'>t''\end{cases}$$

$$\frac{\partial}{\partial t'}G(t',t'')=\begin{cases}\frac{E}{t''} & t'\leq t''\\ \frac{E}{t''-t} & t'>t''\end{cases}$$

$$=\frac{E}{t''}+\Theta(t'-t'')\left(\frac{E}{t''-t}-\frac{E}{t''}\right)$$

$$\frac{\partial^2}{\partial t'^2}G(t',t'')=\delta(t'-t'')E\left(\frac{1}{t''-t}-\frac{1}{t''}\right)\overset{!}{=}-\frac{\hbar}{m}\delta(t'-t'')$$

We thus need

$$E=-\frac{\hbar}{m}\frac{1}{\frac{1}{t''-t}-\frac{1}{t''}}=-\frac{\hbar}{m}\frac{1}{\frac{t''-(t''-t)}{(t''-t)t''}}=-\frac{\hbar}{m}\frac{(t''-t)t''}{t}$$

$$G(t',t'')=-\frac{\hbar}{m}\begin{cases}t'(t''-t) & t'\leq t''\\ t''(t'-t) & t'>t''\end{cases}$$

which agrees with

$$G(t',t'')=-\frac{\hbar}{m\omega\sin\omega t}\begin{cases}\sin\omega t'\,\sin\omega(t''-t) & t'\leq t''\\ \sin\omega t''\,\sin\omega(t'-t) & t'>t''\end{cases}$$

for $\omega\to 0$