Advanced Quantum Theory

Problems

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The homework for Advanced Quantum Theory will be set from this handout, from Thursdays to Thursday in the following week at 4pm.

This document also contains past exam questions for the course, and a few selected solutions to questions that I am not planning to set as homework (e.g. for students who want to see additional solved problems before starting their homework).

1 Basics

1.1 Stationarity of the action in Hamiltonian mechanics

The analogue of the action in Hamiltonian mechanics is

$$S[\boldsymbol{q},\boldsymbol{p}] = \int_{t_1}^{t_2} (\boldsymbol{p}(t')\cdot\dot{\boldsymbol{q}}(t') - H(\boldsymbol{q}(t'),\boldsymbol{p}(t'))dt'$$

where $H(\boldsymbol{q},\boldsymbol{p})$ is the Hamiltonian of the system. Using the Euler-Lagrange equations from Mechanics 2/23 determine the conditions under which $S[\boldsymbol{q},\boldsymbol{p}]$ is stationary w.r.t. variations of the functions $\boldsymbol{q}(t)$ and $\boldsymbol{p}(t)$ that preserve the boundary conditions at t_1 and t_2.

1.2 A representation of the delta function

Show that

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ipx/\hbar}dp = \delta(x).$$

Possible ways to solve this problem are

(a) Consider

$$\frac{1}{2\pi\hbar}\lim_{a\to 0}\int_{-\infty}^{\infty} e^{ipx/\hbar - ap^2}dp$$

and use that

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon}}e^{-y^2/\epsilon} = \delta(y).$$

(b) Use that subsequent application of the Fourier transform and the inverse Fourier transform to a function returns the function we started from.

2 Feynman path integral

2.1 Path integral in phase space

Show that the propagator of a quantum system can be written as

$$K(\boldsymbol{r}_f, \boldsymbol{r}_0, t) = \int \mathcal{D}[\boldsymbol{r}]\mathcal{D}[\boldsymbol{p}] \exp\left(\frac{i}{\hbar}\int_0^t (\boldsymbol{p}(t')\cdot\dot{\boldsymbol{r}}(t') - H(\boldsymbol{r}(t'), \boldsymbol{p}(t'))dt'\right)$$

Here the integral is taken over all functions $\boldsymbol{r}(t')$ with $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and $\boldsymbol{r}(t) = \boldsymbol{r}_f$, and over all functions $\boldsymbol{p}(t')$ regardless of boundary conditions.
To solve this problem use the result

$$K(\boldsymbol{r}_f, \boldsymbol{r}_0, t) = \int d^n r_1 \ldots \int d^n r_{N-1} \prod_{j=0}^{N-1} \langle \boldsymbol{r}_{j+1}| e^{-\frac{i}{\hbar}\hat{H}\tau} |\boldsymbol{r}_j\rangle$$

from the lecture (with $\boldsymbol{r}_N := \boldsymbol{r}_f$), as well as an expression for $\langle \boldsymbol{r}_j | e^{-\frac{i}{\hbar}\hat{H}\tau}|\boldsymbol{r}_{j-1}\rangle$ where the integral over the momentum has not been evaluated, such that momenta for the different time steps remain as integration variables in the final result.

Note that the integration measure will be different from the $D[\boldsymbol{r}]$ appearing in the position-space path integral. Your calculation should give the appropriate meaning of $\int \mathcal{D}[\boldsymbol{r}]\mathcal{D}[\boldsymbol{p}]\ldots$.

2.2 Propagator for the harmonic oscillator

In the lecture we determined the propagator for the harmonic oscillator. Check explicitly that the obtained result satisfies the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_f^2} + \frac{1}{2}m\omega^2 x_f^2\right) K(x_f, x_0, t) = i\hbar\frac{\partial}{\partial t}K(x_f, x_0, t)$$

and that it satisfies $K(x_f, x_0, 0) = \langle x_f | x_0\rangle = \delta(x_f - x_0)$. It may be helpful to use

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{i\pi\epsilon}} e^{iy^2/\epsilon} = \delta(y).$$

2.3 Propagator for a free particle

Evaluate the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Lagrangian $L = \frac{1}{2}m\dot{x}^2$.

Hints: As for the harmonic oscillator, split $x(t')$ into the classical solution $x_{\text{cl}}(t')$ and the deviation $\delta x(t')$ from this solution. Then work with the discretised version of the position-space path integral, and show that the action $S[\delta x]$ can be written as

$$S[\delta x] = \delta\boldsymbol{x}\cdot A\delta\boldsymbol{x}$$

where $\delta\boldsymbol{x}$ is a vector whose components are the values of δx at the time steps and A is a real symmetric matrix. Moreover you can use that for such a matrix we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(i\delta\boldsymbol{x}\cdot A\delta\boldsymbol{x}\right) d\delta x_1 d\delta x_2 \ldots d\delta x_\nu = \left(\frac{(i\pi)^\nu}{\det A}\right)^{1/2}.$$

(how do we have to choose ν for our problem?) and that the $\nu \times \nu$ matrix of the form

$$B_\nu = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

has the determinant $\det B_\nu = \nu + 1$.

2.4 Free particle in Hamiltonian mechanics

Evaluate the Hamiltonian version of the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Hamiltonian $H = \frac{p^2}{2m}$.

2.5 Short-time propagator [2019 exam]

We consider a one dimensional system whose potential U is an *arbitrary* function of the position x and whose kinetic energy T is an *arbitrary* function of the momentum p. Show that for short times t the corresponding propagator can be approximated by

$$K(x_f, x_0, t) \approx \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \exp\left(\frac{i}{\hbar}\left[p\frac{x_f - x_0}{t} - T(p) - U(x_0)\right] t\right)$$

where the two sides agree neglecting corrections of order t^2 and higher.

2.6 Propagator in momentum basis [2021 exam]

(a) We consider a one-dimensional system with the Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + U(\hat{x})$$

where $\hat{p}$ is the momentum operator and $\hat{x}$ is the position operator. Derive an expression for

$$\langle p_f | e^{-\frac{i}{\hbar}\hat{H}t} | p_0 \rangle$$

where $|p_0\rangle$ and $|p_f\rangle$ are eigenstates of the momentum with eigenvalues p_0 and p_f. Your expression should be valid for *short times* t, neglecting terms of order t^2 and higher, and it should involve a single integral over positions. You can use that the wavefunction associated to a momentum eigenstate $|p\rangle$ is $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

(b) Use the results of (a) to derive an expression for $\langle p_f | e^{-\frac{i}{\hbar}\hat{H}t} | p_0 \rangle$ that holds for arbitrary times and involves a path integral in phase space.