problem based on 2015, 3

$$\left\langle x(t_1)\,x(t_2)\,x(t_3)\,\exp\left(-\frac{i\varepsilon}{\hbar}\int_0^t x(t')^3\,dt'\right)\right\rangle$$

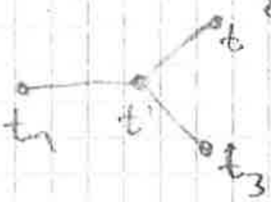
$$= \Big\langle x(t_1)\,x(t_2)\,x(t_3)\Big(\cancel{1} - \frac{i\varepsilon}{\hbar}\int_0^t x(t')^3\,dt'$$

$$+ \frac{1}{2}\left(\frac{i\varepsilon}{\hbar}\right)^2 \cancel{\int_0^t x(t')^3\,dt' \int_0^t x(t'')^3\,dt''} + O(\varepsilon^3)\Big)\Big\rangle$$

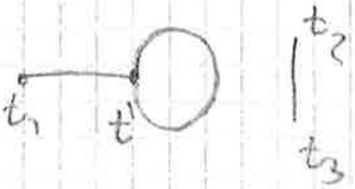
$$= -\frac{i\varepsilon}{\hbar}\int_0^t dt' \left\langle x(t_1)\,x(t_2)\,x(t_3)\,x(t')^3\right\rangle + O(\varepsilon^3)$$

Here the crossed-out terms involve an odd overall number of factors, hence their averages vanish.

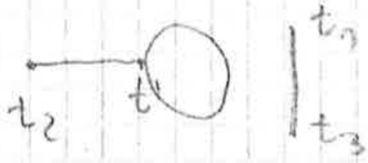
The contributing Feynman diagrams are



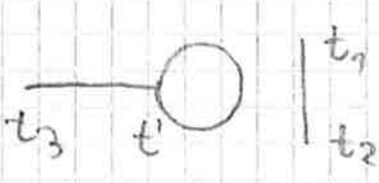
multiplicity: 6
(3 choices for $x(t')$ contracted with $x(t_1)$,
2 choices for $x(t')$ contracted with $x(t_2)$)



multiplicity: 3
(3 choices for $x(t')$ contracted with $x(t_1)$)



mult: 3



mult: 3

Hence we obtain

$$-\frac{i\varepsilon}{\hbar}\Big(6\int_0^t dt'\; iG(t_1,t')\,iG(t_2,t')\,iG(t_3,t')$$

$$+ 3\int_0^t dt'\; iG(t_1,t')\,iG(t',t')\,iG(t_2,t_3)$$

$$+ 3\int_0^t dt'\; iG(t_2,t')\,iG(t',t')\,iG(t_1,t_3)$$

$$+ 3\int_0^t dt'\; iG(t_3,t')\,iG(t',t')\,iG(t_1,t_2)\Big) + O(\varepsilon^3)$$

2015, 2(a)

$$\hat{H} = -a_1^+ a_2 - a_2^+ a_1 + \frac{U}{2}\left(\hat{n}_1(\hat{n}_1 - 1) - \hat{n}_2(\hat{n}_2 - 1)\right)$$

evaluate $\hat{H}|n_1, n_2\rangle$

$$\hat{H}|n_1, n_2\rangle = \underbrace{-a_1^+ \sqrt{n_2}\,|n_1, n_2 - 1\rangle}_{=\sqrt{(n_1+1)n_2}\,|n_1+1,\, n_2-1\rangle} \underbrace{- a_2^+ \sqrt{n_1}\,|n_1 - 1, n_2\rangle}_{=\sqrt{n_1(n_2+1)}\,|n_1-1,\, n_2+1\rangle}$$

$$+ \frac{U}{2}\left(n_1(n_1 - 1) - n_2(n_2 - 1)\right)|n_1, n_2\rangle$$

energies for the case that there are two particles

use

$$\hat{H}|2,0\rangle = -\sqrt{2}\,|1,1\rangle + U|0,2\rangle$$

$$\hat{H}|1,1\rangle = -\sqrt{2}\,|2,0\rangle - \sqrt{2}\,|0,2\rangle$$

$$\hat{H}|0,2\rangle = -\sqrt{2}\,|1,1\rangle - U|0,2\rangle$$

see model solution for evaluation of energy levels

now consider N particles and write down a matrix that have the same eigenvalues as $\hat{H}$ for this case

basis states: $|n, N-n\rangle \qquad n = 0 \dots N$

The matrix elements are

$$H_{mn} = \langle m|N-m|\, \hat{H}\, |n, N-n\rangle$$

$$= \langle m|N-m|\Big[-\sqrt{(n+1)(N-n)}\,|n+1, N-n-1\rangle$$

$$- \sqrt{n(N-n+1)}\,|n-1, N-n+1\rangle$$

$$+ \frac{U}{2}\left(n(n-1) - (N-n)(N-n-1)\right)|n, N-n\rangle\Big]$$

$$= \delta_{m,n+1}\left(-\sqrt{(n+1)(N-n)}\right)$$

$$+ \delta_{m,n-1}\left(-\sqrt{n(N-n+1)}\right)$$

$$+ \delta_{m,n}\,\frac{U}{2}\left(n(n-1) - (N-n)(N-n-1)\right)$$

i.e. the matrix is

$$\begin{pmatrix} H_{00} & H_{01} & & & \\ H_{10} & H_{11} & H_{12} & & \\ & H_{21} & H_{22} & \ddots & \\ & & \ddots & \ddots & \\ & & & & H_{NN} \end{pmatrix}$$

with

$$H_{n+1,n} = -\sqrt{(n+1)(N-n)}$$

$$H_{n-1,n} = -\sqrt{n(N-n+1)}$$

$$H_{n,n} = \frac{U}{2}\left(n(n-1) - (N-n)(N-n-1)\right)$$