Advanced Quantum Theory

Problem sheet 1
Path integrals

Please hand in questions 2(a) and 5 by Thursday 8 February 4pm. Solutions can be handed in during the lecture, or in the box in the Maths building, or by email to sebastian.muller@bristol.ac.uk

1. *A representation of the delta function*
   Show that
   $$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ipx/\hbar} dp = \delta(x).$$
   Possible ways to solve this problem are
   (a) Consider
   $$\frac{1}{2\pi\hbar}\lim_{a\to 0}\int_{-\infty}^{\infty} e^{ipx/\hbar - ap^2} dp$$
   and use that
   $$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon}} e^{-y^2/\epsilon} = \delta(y).$$
   (b) Use that subsequent application of the Fourier transform and the inverse Fourier transform to a function returns the function we started from.

2. *Path integral in phase space*
   (a) Show that the propagator of a quantum system can be written as
   $$K(\boldsymbol{r}_f, \boldsymbol{r}_0, t) = \int \mathcal{D}[\boldsymbol{r}]\mathcal{D}[\boldsymbol{p}] \exp\left(\frac{i}{\hbar}\int_0^t (\boldsymbol{p}(t')\cdot\dot{\boldsymbol{r}}(t') - H(\boldsymbol{r}(t'), \boldsymbol{p}(t'))dt'\right)$$
   Here the integral is taken over all functions $\boldsymbol{r}(t')$ with $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and $\boldsymbol{r}(t) = \boldsymbol{r}_f$, and over all functions $\boldsymbol{p}(t')$ regardless of boundary conditions.
   Proceed as for the position-space path integral in the lecture. Do not perform the integral over momenta involved in one of the steps, but instead keep the momenta so that they show up in the final result. Note that the integration measure will be different from the $D[\boldsymbol{r}]$ appearing in the position-space path integral. Your calculation should give the appropriate meaning of $\int \mathcal{D}[\boldsymbol{r}]\mathcal{D}[\boldsymbol{p}]\ldots$.
   (b) Derive the same result as in (a) using a different method: Splitting the time range from 0 to t into N steps of duration $\tau = \frac{t}{N}$ and using the results for the short time propagator we can write
   $$e^{-\frac{i}{\hbar}\hat{H}t} \approx \left(e^{-\frac{i}{\hbar}\hat{T}\tau} e^{-\frac{i}{\hbar}\hat{U}\tau}\right)^N.$$
   Now insert the resolution of the identity
   $$1 = \int d^n r |\boldsymbol{r}\rangle\langle\boldsymbol{r}|$$

after each exponential involving the potential, and the resolution of the identity in terms of momentum eigenstates $|\boldsymbol{p}\rangle$,

$$1 = \int d^n p |\boldsymbol{p}\rangle\langle\boldsymbol{p}|,$$

after each exponential involving the kinetic energy. You can take as given that the scalar product of position and momentum eigenstates is

$$\langle\boldsymbol{r}|\boldsymbol{p}\rangle = \frac{1}{(2\pi\hbar)^{n/2}} e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}}.$$

Show that this approach leads to the same phase-space path integral as in (a).

3. *Stationarity of the action in Hamiltonian mechanics*
The analogue of the action in Hamiltonian mechanics is

$$S[\boldsymbol{r},\boldsymbol{p}] = \int_{t_1}^{t_2} (\boldsymbol{p}(t')\cdot\dot{\boldsymbol{r}}(t') - H(\boldsymbol{r}(t'),\boldsymbol{p}(t'))dt'$$

where $H(\boldsymbol{r},\boldsymbol{p})$ is the Hamiltonian of the system. Using the Euler-Lagrange equations from Mechanics 2/23 determine the conditions under which $S[\boldsymbol{r},\boldsymbol{p}]$ is stationary w.r.t. variations of the functions $\boldsymbol{r}(t)$ and $\boldsymbol{p}(t)$ that preserve the boundary conditions at t_1 and t_2.

4. *Propagator for the harmonic oscillator*
In the lecture we determined the propagator for the harmonic oscillator. Check explicitly that the obtained result satisfies the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_f^2} + \frac{1}{2}m\omega^2 x_f^2\right) K(x_f,x_0,t) = i\hbar\frac{\partial}{\partial t}K(x_f,x_0,t)$$

and that it satisfies $K(x_f,x_0,0) = \langle x_f|x_0\rangle = \delta(x_f - x_0)$. It may be helpful to use

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi\epsilon}}e^{iy^2/\epsilon} = \delta(y).$$

5. *Propagator for a free particle*
Evaluate the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Lagrangian $L = \frac{1}{2}m\dot{x}^2$.

Hints: As for the harmonic oscillator, split $x(t')$ into the classical solution $x_{\text{cl}}(t')$ and the deviation $\delta x(t')$ from this solution. Then work with the discretised version of the position-space path integral, and to use that for an $\nu\times\nu$ real symmetric matrix A we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(i\sum_{j=1}^{\nu}\sum_{k=1}^{\nu}y_j A_{jk} y_k\right)dy_1 dy_2\ldots dy_\nu = \left(\frac{(i\pi)^\nu}{\det A}\right)^{1/2}.$$

Moreover the $\nu\times\nu$ matrix of the form

$$B_\nu = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

has the determinant $\det B_\nu = \nu + 1$.

6. *Free particle in Hamiltonian mechanics*

   Evaluate the Hamiltonian version of the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Hamiltonian $H = \frac{p^2}{2m}$.

Drop-in session: Tue 15:30-16:30 (Howard House 4.08)