Advanced Quantum Theory

Problem sheet 2
Continuum limit, Perturbation theory

Set: 1, 4
Due: Monday 13 February after the lecture (Solutions can be handed in during the lecture, or in the box in the Maths building, or by email to sebastian.muller@bristol.ac.uk)

1. *Elastic chain*

   We consider a chain of $N+1$ labelled by indices $i = 0, \ldots, N$. As in the lecture masses with neighbouring indices are connected by a spring with spring constant k and natural length zero. In contrast to the lecture the positions of the masses are taken as two-dimensional. The position of the i-th mass is denoted by $\boldsymbol{\Phi}_i = (\Phi_{i1}, \Phi_{i2})$. The mass $i = 0$ is fixed at $\Phi_{01} = \Phi_{02} = 0$, the final mass with index $i = N$ is fixed at $\Phi_{N1} = C, \Phi_{N2} = 0$. The masses have a kinetic energy, a potential energy due to the tension of the springs, and also a gravitational energy $mg\Phi_{i2}$ for each mass.

   (a) Write the propagator of the system as a path integral. (You don't need to evaluate the path integrals in this question.)

   (b) Then take the limit $N \to \infty$ of this path integral, choosing the equilibrium values of the positions Φ_{i1} as a continuous parameter replacing the index i.

   (c) Write down a path integral analogous to (b) for the matrix elements of $e^{-\beta \hat{H}}$.

2. *Euler-Lagrange equations for the continuous case*

   (a) Consider a functional acting on functions $\boldsymbol{r}(x, t')$ where x and t' are real numbers and $\boldsymbol{r}$ is in $\mathbb{R}^n$. The functional is defined through the integral

   $$S[\boldsymbol{r}] = \int_0^t dt' \int_0^C dx\ \mathcal{L}(\boldsymbol{r}(x,t'), \boldsymbol{r}'(x,t'), \dot{\boldsymbol{r}}(x,t'))$$

   where $\boldsymbol{r}' = \frac{\partial \boldsymbol{r}}{\partial x}$, $\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial t'}$. Show that $S[\boldsymbol{r}]$ is stationary w.r.t. variations of $\boldsymbol{r}(x, t')$ (that preserve the boundary conditions at $x = 0, C$ and $t' = 0, t$ if

   $$\frac{\partial \mathcal{L}}{\partial \boldsymbol{r}} = \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}'} + \frac{\partial}{\partial t'} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{r}}} \ .$$

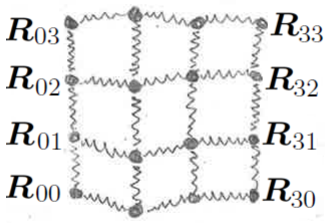
   For the proof you can proceed similarly to the proof of the Euler-Lagrange equations in Mechanics 2/23.

   (b) Using (a) give an example for a function $\Phi(x, t')$ for which the action of the elastic chain in chapter 2.4 is stationary.

   (c) Determine the Euclidian action for the elastic chain, and give an example for a function $\Psi(x, t')$ for which this Euclidian action is stationary.

3. *A two dimensional grid with springs*

   (Modified from the 2015/16 exam.) Consider a two-dimensional grid of $(N+1)^2$ masses m connected by springs (as sketched in the figure for the case $N = 3$). The masses are numbered by integers $i = 0 \dots N$ and $j = 0 \dots N$ and their positions are denoted by $\boldsymbol{R}_{ij} = \begin{pmatrix} X_{ij} \\ Y_{ij} \end{pmatrix}$. The masses whose indices i differ by 1 are connected by a spring, and the same holds for masses whose indices j differ by 1. The springs have the spring constant k and the natural length 0. The positions of the masses with $i = 0$ and $i = N$ are fixed according to $\boldsymbol{R}_{0j} = \begin{pmatrix} 0 \\ \frac{j}{N} \end{pmatrix}$, $\boldsymbol{R}_{Lj} = \begin{pmatrix} 1 \\ \frac{j}{N} \end{pmatrix}$, and the remaining masses are allowed to move freely.



   (a) Write down the Lagrangian and the action of the system. You are not asked to take into account gravity.

   (b) Show that in the continuum limit $L \to \infty$, $\boldsymbol{R}_{ij}(t')$ can be replaced by a field $\boldsymbol{R}(x, y, t')$ and the Lagrangian from (b)i can be written as a multiple integral over a Lagrangian density that depends on $\boldsymbol{R}(x, y, t')$ and/or its first derivatives w.r.t. x, y, and t'.

4. *Feynman's trick*

   Express the integrals

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2/2} dx$$

   (where n is integer) through derivatives of

$$\int_{-\infty}^{\infty} e^{-ax^2/2} dx$$

   w.r.t. a and use this to evaluate these integrals. This idea is similar to (but slightly less powerful than) the method we will use to evaluate these integrals in the lecture.

Office hours: Thu 10:30-11:30, Fri 10:30-11:30 (Howard House 4.08)