# Advanced Quantum Theory

**Problem sheet 3**
**Wick's theorem**

Please hand in questions 2a+b and 3a by Thursday 22 February 4pm. Solutions can be handed in during the lecture, or in the box in the Maths building, or by email to sebastian.muller@bristol.ac.uk

1. *Complex Gaussian integral*

   (a) Evaluate the integral
   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a|z|^2}\, d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z$$
   (for $a>0$) and, using this result, determine
   $$c \equiv \int_{\mathbb{C}^n} e^{-\boldsymbol{z}^\dagger A \boldsymbol{z}} \prod_{k=1}^{n} d\,\mathrm{Re}\, z_k\, d\,\mathrm{Im}\, z_k$$
   where $z\in\mathbb{C}^n$ and $A$ is a hermitian (self-adjoint) positive definite $n\times n$ matrix.

   (b) We now consider the average
   $$\langle \ldots \rangle \equiv \frac{1}{c}\int_{\mathbb{C}^n} e^{-\boldsymbol{z}^\dagger A \boldsymbol{z}} \ldots \prod_{k=1}^{n} d\,\mathrm{Re}\, z_k\, d\,\mathrm{Im}\, z_k .$$
   Show that $\langle z_k z_{k'}\rangle = \langle z_k^* z_{k'}^*\rangle = 0$ and evaluate $z_k z_{k'}^*$. This can be done by considering a generating function $\langle e^{\boldsymbol{j}^\dagger \boldsymbol{z} + \boldsymbol{z}^\dagger \boldsymbol{j}}\rangle$ and taking derivatives w.r.t. this generating function treating $j_k$ and $j_k^*$ as independent variables.

2. *Wick's theorem*

   Use Wick's theorem to evaluate the following integrals:

   (a) $\int (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} dxdydz$

   (b) $\int x_i x_j e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \epsilon\sum_k x_k^4} d^n x$ neglecting terms of order $\epsilon^2$ and higher, for real symmetric positive definite $n\times n$ matrices and $i,j$ being integers between 1 and $n$

   (c) $\int e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \epsilon(x_1^4+x_2^4)} d^2 x$ neglecting terms of order $\epsilon^3$ and higher, for $\boldsymbol{x}\in\mathbb{R}^2$ and real symmetric positive definite $2\times 2$ matrices $A$

   (d) $\int e^{\frac{i}{2}\boldsymbol{x}^T A\boldsymbol{x} + i\epsilon\sum_k x_k^3} d^n x$ neglecting terms of order $\epsilon^3$ and higher, for real symmetric invertible $n\times n$ matrices $A$

3. *Matrix integrals*

   We are considering real symmetric $2\times 2$ matrices $H$ and we define
   $$\langle \ldots \rangle \equiv c^{-1}\int e^{-\mathrm{tr} H^2} \ldots dH_{11}\, dH_{12}\, dH_{22}$$
   where
   $$c \equiv \int e^{-\mathrm{tr} H^2} dH_{11}\, dH_{12}\, dH_{22}\, .$$
   Use Wick's theorem to evaluate

(a) $\langle \det(H - E) \rangle$ where $E$ is a real number

(b) $\langle H_{\alpha\beta} H_{\gamma\delta} \rangle$

(c) Write down the possible contractions for

$$\langle \operatorname{tr} H^4 \rangle = \sum_{\alpha\beta\gamma\delta} \langle H_{\alpha\beta} H_{\beta\gamma} H_{\gamma\delta} H_{\delta\alpha} \rangle.$$

For each way of drawing contraction lines, which conditions do the summation variables $\alpha, \beta, \gamma, \delta$ have to satisfy to give nonvanishing contributions to the average?

4. *Integral kernel*

In the lecture we looked for the kernel $G(t', t'')$ subject to the conditions

$$\left(-\frac{m}{\hbar}\right) \left(\frac{\partial^2}{\partial t'^2} + \omega^2\right) G(t', t'') = \delta(t' - t'')$$

and

$$G(0, t'') = G(t, t'') = 0.$$

We gave a formula for this kernel and proved it. Now imagine that you don't know this result and determine $G(t', t'')$ from the above formulas by calculation. Consider

(a) the case $\omega = 0$ corresponding to a free particle, and

(b) the case of arbitrary $\omega \neq 0$.

Drop in session: Thu 10:30-11:30, Fri 10:30-11:30 (Howard House 4.08)

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