Advanced Quantum Theory

Problem sheet 3
Wick's theorem

Set: 1a, 2a+b, 3a
Due: Monday 20 February (Solutions can be handed in during the lecture, or in the box in the Maths building, or by email to sebastian.muller@bristol.ac.uk)

1. *Complex Gaussian integral*

   (a) Evaluate the integral
   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a|z|^2} d\,\mathrm{Re}\,z\,d\,\mathrm{Im}\,z$$
   (for $a>0$) and, using this result, determine
   $$c \equiv \int_{\mathbb{C}^n} e^{-\boldsymbol{z}^\dagger A\boldsymbol{z}} \prod_{k=1}^{n} d\,\mathrm{Re}\,z_k\,d\,\mathrm{Im}\,z_k$$
   where $z\in\mathbb{C}^n$ and A is a hermitian (self-adjoint) positive definite $n\times n$ matrix.

   (b) We now consider the average
   $$\langle\ldots\rangle \equiv \frac{1}{c}\int_{\mathbb{C}^n} e^{-\boldsymbol{z}^\dagger A\boldsymbol{z}}\ldots \prod_{k=1}^{n} d\,\mathrm{Re}\,z_k\,d\,\mathrm{Im}\,z_k .$$
   Show that $\langle z_k z_{k'}\rangle = \langle z_k^* z_{k'}^*\rangle = 0$ and evaluate $z_k z_{k'}^*$. This can be done by considering a generating function $\langle e^{\boldsymbol{j}^\dagger \boldsymbol{z}+\boldsymbol{z}^\dagger \boldsymbol{j}}\rangle$ and taking derivatives w.r.t. this generating function treating j_k and j_k^* as independent variables.

2. *Wick's theorem*

   Use Wick's theorem to evaluate the following integrals:

   (a) $\int (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} dxdydz$

   (b) $\int x_i x_j e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x}-\epsilon\sum_k x_k^4} d^n x$ neglecting terms of order ϵ^2 and higher, for real symmetric positive definite $n\times n$ matrices and i,j being integers between 1 and n

   (c) $\int e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x}-\epsilon(x_1^4+x_2^4)} d^2 x$ neglecting terms of order ϵ^3 and higher, for $\boldsymbol{x}\in\mathbb{R}^2$ and real symmetric positive definite 2×2 matrices A

   (d) $\int e^{\frac{i}{2}\boldsymbol{x}^T A\boldsymbol{x}+i\epsilon\sum_k x_k^3} d^n x$ neglecting terms of order ϵ^3 and higher, for real symmetric invertible $n\times n$ matrices A

3. *Matrix integrals*

   We are considering real symmetric 2×2 matrices H and we define
   $$\langle\ldots\rangle \equiv c^{-1}\int e^{-\mathrm{tr}H^2}\ldots dH_{11}\,dH_{12}\,dH_{22}$$
   where
   $$c \equiv \int e^{-\mathrm{tr}H^2} dH_{11}\,dH_{12}\,dH_{22}\,.$$
   Use Wick's theorem to evaluate

(a) $\langle \det(H - E) \rangle$ where E is a real number

(b) $\langle H_{\alpha\beta} H_{\gamma\delta} \rangle$

(c) Write down the possible contractions for

$$\langle \mathrm{tr}\, H^4 \rangle = \sum_{\alpha\beta\gamma\delta} \langle H_{\alpha\beta} H_{\beta\gamma} H_{\gamma\delta} H_{\delta\alpha} \rangle.$$

For each way of drawing contraction lines, which conditions do the summation variables $\alpha, \beta, \gamma, \delta$ have to satisfy to give nonvanishing contributions to the average?

Drop in session: Thu 10:30-11:30, Fri 10:30-11:30 (Howard House 4.08)