Advanced Quantum Theory

Problem sheet 5
Second quantisation

Set: 1, 2

Due: Monday 6 March 4pm (Solutions can be handed in during the lecture, or in the box in the Maths building, or by email to sebastian.muller@bristol.ac.uk)

1. *States for systems with three particles*

   (a) Consider a three-particle system in which we know that the three (different) single-particle states $|i\rangle$, $|j\rangle$, and $|k\rangle$ are occupied by one particle each. Write this state as a linear combination of suitable basis states. Note that at this stage we do not yet require the particles to be indistinguishable.

   Then assume that the particles are indeed indistinguishable and bosonic. Which relation does this imply for the coefficients in your linear combination? Show that this relation is equivalent to the result given in the lecture.

   Then do the same for fermionic particles.

   (b) Your result in (a) should give you the state up to a normalisation factor. Determine this factor.

   (c) Proceed as in (a) and (b) for a bosonic system where the state $|i\rangle$ is occupied twice, and the state $|j\rangle \neq |i\rangle$ is occupied once.

2. *Bose-Hubbard model*
   We consider a system allowing for an arbitrary number of indistinguishable bosonic particles. The system has two sites, and the operators $a_i^\dagger$ and a_i ($i = 1, 2$) are the creation and annihilation operators for particles on these sites. The Hamiltonian for the system is

$$\hat{H} = -a_1^\dagger a_2 - a_2^\dagger a_1 + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1)$$

   where $\hat{n}_i = a_i^\dagger a_i$. Here the first two terms are related to particles hopping between the two sites, and the final term represents an interaction potential arising if particles share the same site. This system is a variant of the Bose-Hubbard model which is widely used in condensed matter theory.

   (a) Show that applying this Hamiltonian to a state does not change the overall number of particles involved in that state. (To show this you can either apply the Hamiltonian to a state in occupation number representation, or you can show that it commutes with the operator giving the overall number of particles.)

   (b) Now we are interested specifically in the case that our system contains two particles. Write down the basis states of the system with two particles.

   (c) Investigate how the Hamiltonian acts on the basis states determined in (b). Use your result to represent the Hamiltonian (now restricted to the case of two particles) in matrix form and determine its eigenvalues.

3. *Relation between a_i and $a_i^\dagger$*
   Show that if $|\Psi\rangle$ and $|\Phi\rangle$ are two states with arbitrarily many indistinguishable bosonic particles and a_i, $a_i^\dagger$ are creation and annihilation operators, we have

$$\langle\Phi|a_i|\Psi\rangle = \langle\Psi|a_i^\dagger|\Phi\rangle^*$$

   i.e. a_i and $a_i^\dagger$ are adjoint operators. In your proof you can use that the scalar product of states with different particle numbers vanishes by definition.

4. *A state*

   For a system with indistinguishable bosonic particles, determine the state

$$\prod_i \frac{(a_i^\dagger)^{m_i}}{\sqrt{m_i!}}|0\rangle$$

   in occupation number representation. Here $|0\rangle$ is the vacuum, and m_i are integers.

Office hours: Thu 10:30-11:30, Fri 10:30-11:30 (Howard House 4.08)