Advanced Quantum Theory
Solutions for problem sheet 5

①

(a) The state has to be a linear combination

$$|\psi\rangle = c_{ijk}\,|i\rangle|j\rangle|k\rangle + c_{ikj}\,|i\rangle|k\rangle|j\rangle + c_{jik}\,|j\rangle|i\rangle|k\rangle + c_{jki}\,|j\rangle|k\rangle|i\rangle + c_{kij}\,|k\rangle|i\rangle|j\rangle + c_{kji}\,|k\rangle|j\rangle|i\rangle$$

(Anti-) symmetry w.r.t. interchanging the first two particles implies (here the upper signs are for bosons and the lower ones for fermions)

$$|\psi\rangle = \pm c_{ijk}\,|j\rangle|i\rangle|k\rangle \pm c_{ikj}\,|k\rangle|i\rangle|j\rangle \pm c_{jik}\,|i\rangle|j\rangle|k\rangle \pm c_{jki}\,|k\rangle|j\rangle|i\rangle \pm c_{kij}\,|i\rangle|k\rangle|j\rangle \pm c_{kji}\,|j\rangle|k\rangle|i\rangle$$

Comparing coefficients gives

$$c_{ijk} = \pm c_{jik} \qquad c_{ikj} = \pm c_{kij} \qquad c_{jki} = \pm c_{kji} \quad (*)$$

(Anti-) symmetry w.r.t. exchanging the first and third particle implies

$$|\psi\rangle = \pm c_{ijk}\,|i\rangle|k\rangle|j\rangle \pm c_{ikj}\,|i\rangle|j\rangle|k\rangle \pm c_{jik}\,|j\rangle|k\rangle|i\rangle \pm c_{jki}\,|j\rangle|i\rangle|k\rangle \pm c_{kij}\,|k\rangle|j\rangle|i\rangle \pm c_{kji}\,|k\rangle|i\rangle|j\rangle$$

Comparing coefficient gives

$$c_{ijk} = \pm c_{ikj} \qquad c_{jik} = \pm c_{jki} \qquad c_{kij} = \pm c_{kji} \quad (**)$$

(*) and (**) together mean that all coefficients are equal for bosons. This agrees with the lecture. All coefficients will be denoted by c. For fermions

$$c = c_{ijk} \underset{(*)}{=} -c_{jik} \underset{(**)}{=} c_{jki} \underset{(*)}{=} -c_{kji} \underset{(**)}{=} c_{kij} \underset{(*)}{=} c_{ijk}.$$

As in the lecture, negative signs arise if the required number of transpositions is odd.

(b) The states $|i\rangle|j\rangle|k\rangle$ etc are normalised as $|i\rangle$, $|j\rangle$, and $|k\rangle$ are normalised. They are also orthogonal to each other, as the scalar product of $|i\rangle|j\rangle|k\rangle$ and $|i'\rangle|j'\rangle|k'\rangle$ is

$$\int d^n r_1\, d^n r_2\, d^n r_3 \;\; \psi_i^*(\underline{r}_1)\, \psi_j^*(\underline{r}_2)\, \psi_k^*(\underline{r}_3)$$
$$\psi_{i'}(\underline{r}_1)\, \psi_{j'}(\underline{r}_2)\, \psi_{k'}(\underline{r}_3)$$

$$= \delta_{ii'}\, \delta_{jj'}\, \delta_{kk'}$$

due to e.g. $\int d^n r_1\, \psi_i^*(\underline{r}_1)\, \psi_{i'}(\underline{r}_1) = \delta_{ii'}$.

Hence

$$\langle\psi|\psi\rangle = |c_{ijk}|^2\, \big||i\rangle|j\rangle|k\rangle\big|^2 + |c_{ikj}|^2\, \big||i\rangle|k\rangle|j\rangle\big|^2 + \ldots$$
$$= |c_{ijk}|^2 + |c_{ikj}|^2 + \ldots$$
$$= 6\,|c|^2$$

(here $|.|$ is the norm of a state, i.e. $\big||\psi\rangle\big|^2 = \langle\psi|\psi\rangle$)

Normalisation thus implies

$$|c| = \frac{1}{\sqrt{6}}$$

and the simplest choice is

$$c = \frac{1}{\sqrt{6}}$$

(c) The state has to be a linear combination

$$|\psi\rangle = c_{iij}\, |i\rangle|i\rangle|j\rangle + c_{iji}\, |i\rangle|j\rangle|i\rangle + c_{jii}\, |j\rangle|i\rangle|i\rangle$$

Symmetry w.r.t. interchanging the first two particles implies

$$|\psi\rangle = c_{iij}\, |i\rangle|i\rangle|j\rangle + c_{iji}\, |j\rangle|i\rangle|i\rangle + c_{jii}\, |i\rangle|j\rangle|i\rangle$$

Comparing coefficients gives

$$c_{iji} = c_{jii}.$$

Symmetry w.r.t. interchanging the 2nd and 3rd particle implies

$$|\psi\rangle = c_{iij}\, |i\rangle|j\rangle|i\rangle + c_{iji}\, |j\rangle|i\rangle|i\rangle + c_{jii}\, |i\rangle|i\rangle|j\rangle$$

Comparing coefficients gives

$$c_{iji} = c_{jii}$$

Hence all coefficients are equal and will be called c. The same argument as in (b) implies

$$\langle\psi|\psi\rangle = |c_{iij}|^2 + |c_{iji}|^2 + |c_{jii}|^2$$
$$= 3|c|^2 \overset{!}{=} 1$$

with the simplest solution

$$c = \frac{1}{\sqrt{3}}$$

(The result could also have been written as in (a) with $i \to i$, $j \to i$, $k \to j$. Then each summand would have appeared twice and the normalisation factor could have been written as $\frac{1}{\sqrt{12}}$, agreeing with the $\frac{1}{\sqrt{N! \prod_k n_k!}}$ stated in the lecture, here $N=3$, $n_i = 2$, $n_j = 1$)

② (a) Solution I

(i) We have

$$\hat{H}\,|n_1, n_2\rangle$$

$$= \left(-a_1^\dagger a_2 - a_2^\dagger a_1 + \frac{1}{2}\sum_i \hat{n}_i(\hat{n}_i - 1)\right)|n_1, n_2\rangle$$

$$= -a_1^\dagger \sqrt{n_2}\,|n_1, n_2 - 1\rangle$$
$$\quad - a_2^\dagger \sqrt{n_1}\,|n_1 - 1, n_2\rangle$$
$$\quad + \frac{1}{2}\left(n_1(n_1-1) + n_2(n_2-1)\right)|n_1, n_2\rangle$$

$$= -\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1, n_2-1\rangle$$
$$\quad - \sqrt{n_2+1}\sqrt{n_1}\,|n_1-1, n_2+1\rangle$$
$$\quad + \frac{1}{2}\left(n_1(n_1-1) + n_2(n_2-1)\right)|n_1, n_2\rangle$$

which is a linear combination of states with the same overall particle number $n_1 + n_2$.

Solution II

If one uses commutators one has to show

$$[\hat{H}, \hat{n}] = 0$$

where $\hat{n} = \hat{n}_1 + \hat{n}_2$. It is simple to show that $[\hat{n}_i, \hat{n}_j] = 0$. and Hence $[\sum_i \hat{n}_i(\hat{n}_i - 1), \hat{n}]$.

Furthermore we have

$$[-a_1^\dagger a_2, \hat{n}]$$

$$= [-a_1^\dagger a_2, a_1^\dagger a_1 + a_2^\dagger a_2]$$

$$= -a_1^\dagger a_2 a_1^\dagger a_1 + a_1^\dagger a_1 a_1^\dagger a_2$$
$$\quad - a_1^\dagger a_2 a_2^\dagger a_2 + a_2^\dagger a_2 a_1^\dagger a_2$$

$$= -a_1^\dagger a_1^\dagger a_1 a_2 + a_1^\dagger a_1 a_1^\dagger a_2$$
$$\quad - a_1^\dagger a_2 a_2^\dagger a_2 + a_1^\dagger a_2^\dagger a_2 a_2$$

$$= -a_1^\dagger \underbrace{(a_1^\dagger a_1 - a_1 a_1^\dagger)}_{= -1} a_2 - a_1^\dagger \underbrace{(a_2 a_2^\dagger - a_2^\dagger a_2)}_{= 1} a_2$$

$$= 0$$

and similarly (obtained by interchanging $1 \leftrightarrow 2$)

$$[-a_2^+ a_1, \hat{n}] = 0$$

Hence indeed

$$[\hat{H}, \hat{n}] = 0$$

(b) The states with two particles are in occupation number repr.

- $|2,0\rangle$ (conventional repr.: $|1,1\rangle$ as there are two particles in state 1)
- $|1,1\rangle$ (conventional repr: $|1,2\rangle$)
- $|0,2\rangle$ (conventional repr: $|2,2\rangle$)

(c) We have (e.g. by applying the formula given in (a)).

$$\hat{H}|2,0\rangle = -0 - \sqrt{1}\sqrt{2}\,|1,1\rangle + \tfrac{1}{2}\left(2(2-1) + 0(0-1)\right)|2,0\rangle$$
$$= -\sqrt{2}\,|1,1\rangle + |2,0\rangle$$

$$\hat{H}|1,1\rangle = -\sqrt{2}\sqrt{1}\,|2,0\rangle - \sqrt{2}\sqrt{1}\,|0,2\rangle + \tfrac{1}{2}\left(1(1-1) + 1(1-1)\right)|1,1\rangle$$
$$= -\sqrt{2}\,|2,0\rangle - \sqrt{2}\,|0,2\rangle$$

$$\hat{H}|0,2\rangle = -\sqrt{1}\sqrt{2}\,|1,1\rangle - 0 + \tfrac{1}{2}\left(0(0-1) + 2(2-1)\right)|0,2\rangle$$
$$= -\sqrt{2}\,|1,1\rangle + |0,2\rangle$$

i.e. for a state

$$c_{20}|2,0\rangle + c_{11}|1,1\rangle + c_{02}|0,2\rangle = |\psi\rangle$$

we get

$$\hat{H}|\psi\rangle = c_{20}\left(-\sqrt{2}\,|1,1\rangle + |2,0\rangle\right) + c_{11}\left(-\sqrt{2}\,|2,0\rangle - \sqrt{2}\,|0,2\rangle\right) + c_{02}\left(-\sqrt{2}\,|1,1\rangle + |0,2\rangle\right)$$

$$= \underbrace{\left(c_{20} - \sqrt{2}\,c_{11}\right)}_{=d_{20}}|2,0\rangle + \underbrace{\left(-\sqrt{2}\,c_{20} - \sqrt{2}\,c_{02}\right)}_{=d_{11}}|1,1\rangle$$

$$+\underbrace{(-\sqrt{2}c_{11}+c_{02})}_{=d_{02}}|0,2\rangle$$

The coefficients of these states are related by

$$\begin{pmatrix} d_{20} \\ d_{11} \\ d_{02} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\sqrt{2} & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}}_{H} \begin{pmatrix} c_{20} \\ c_{11} \\ c_{02} \end{pmatrix}$$

where H is the matrix representation of the Hamiltonian for 2-particle states. Its eigenvalues are the energy levels for 2-particle states. They are determined by

$$0 = \det(H-E) = \det\begin{pmatrix} 1-E & -\sqrt{2} & 0 \\ -\sqrt{2} & -E & -\sqrt{2} \\ 0 & -\sqrt{2} & 1-E \end{pmatrix}$$

$$= (1-E)^2(-E) - 2(-\sqrt{2})^2(1-E)$$

$$= (1-E)(-E+E^2-4)$$

$$= -(1-E)(E^2-E-4)$$

$$\Leftrightarrow E = 1 \quad \text{or} \quad E = \frac{1}{2} \pm \sqrt{\frac{1}{4}+4} = \frac{1}{2} \pm \sqrt{\frac{17}{4}} = \frac{1}{2}(1 \pm \sqrt{17})$$

(3)

We first consider the case that $|\Phi\rangle$ and $|\Psi\rangle$ are occupation number eigenstates

$$|\Psi\rangle = |n_1, n_2, \ldots, n_i, \ldots\rangle$$
$$|\Phi\rangle = |m_1, m_2 \ldots, m_i, \ldots\rangle$$

The l.h.s. gives

$$\langle\Phi|a_i|\Psi\rangle$$
$$= \langle \ldots m_i \ldots | a_i | \ldots n_i \ldots\rangle$$
$$= \langle \ldots m_i \ldots | \sqrt{n_i} | \ldots n_i - 1 \ldots\rangle$$

The occupation number eigenstates form an orthogonal basis so their scalar product is 1 if the states are identical and 0 otherwise.

This implies

$$\langle\Phi|a_i|\Psi\rangle = \left(\prod_{\substack{j=1 \\ j\neq i}}^{M} \delta_{m_j, n_j}\right) \delta_{m_i, n_i - 1} \sqrt{n_i}$$

On the r.h.s. we obtain

$$\langle\Psi|a_i^\dagger|\Phi\rangle^* = \langle \ldots n_i \ldots | a_i^\dagger | \ldots m_i \ldots\rangle^*$$
$$= \langle \ldots n_i \ldots | \sqrt{m_i + 1} | \ldots m_i + 1 \ldots\rangle^*$$

and using the same result about scalar products

$$\langle\Psi|a_i^\dagger|\Phi\rangle^* = \left(\prod_{\substack{j=1 \\ j\neq i}}^{M} \delta_{n_j, m_j}\right) \underbrace{\delta_{n_i, m_i + 1}}_{\delta_{m_i, n_i - 1}} \underbrace{\sqrt{m_i + 1}}_{= \sqrt{n_i} \text{ if } m_i = n_i - 1}$$

$$= \langle\Phi|a_i|\Psi\rangle$$

A general state can be written as a linear combination of occupation number eigenstates. Writing $\underline{n} = (n_1, n_2, \ldots)$, $\underline{m} = (m_1, m_2, \ldots)$ we can write general states as

$$|\Psi\rangle = \sum_{\underline{n}} c_{\underline{n}} |\underline{n}\rangle$$
$$|\Phi\rangle = \sum_{\underline{m}} d_{\underline{m}} |\underline{m}\rangle \Rightarrow \langle\Phi| = \sum_{\underline{m}} d_{\underline{m}}^* \langle\underline{m}|$$

So using the results for occupation number eigenstates we get

$$\langle \underline{\Phi} | a_i | \underline{\Psi} \rangle^*$$

$$= \sum_{\underline{m}} \sum_{\underline{n}} d^*_{\underline{m}} \, c_n \underbrace{\langle \underline{m} | a_i | \underline{n} \rangle}_{= \langle \underline{n} | a_i^\dagger | \underline{m} \rangle^*}$$

$$= \left(\sum_{\underline{m}} \sum_{\underline{n}} d_{\underline{m}} \, c^*_{\underline{n}} \langle \underline{n} | a_i^\dagger | \underline{m} \rangle \right)^*$$

$$= \left(\left(\sum_n c^*_n \langle \underline{n} | \right) a_i^\dagger \left(\sum_{\underline{m}} d_{\underline{m}} | \underline{m} \rangle \right) \right)^*$$

$$= \left(\langle \underline{\Psi} | \quad a_i^\dagger \quad | \underline{\Psi} \rangle \right)^*$$

as desired

Using $a_i^+ |\ldots n_i \ldots\rangle = \sqrt{n_i+1}\, |\ldots n_i+1 \ldots\rangle$
we get

$$a_i^+ |\ldots n_i=0 \ldots\rangle = |\ldots n_i=1 \ldots\rangle$$

$$a_i^{+2} |\ldots n_i=0 \ldots\rangle = \sqrt{2}\, |\ldots n_i=2 \ldots\rangle$$

etc.

so we expect

$$\frac{(a_i^+)^{m_i}}{\sqrt{m_i!}} |\ldots n_i=0 \ldots\rangle = |\ldots n_i=m_i \ldots\rangle$$

This can be shown by induction. For $m_i=0$ we have

$$\frac{(a_i^+)^0}{\sqrt{0!}} |\ldots n_i=0 \ldots\rangle = |\ldots n_i=0 \ldots\rangle$$

which is in line with the expectation.
Now we assume the expectation is correct for m_i and show that this implies it is correct for m_i+1.
Indeed we have

$$\frac{(a_i^+)^{m_i+1}}{\sqrt{(m_i+1)!}} |\ldots n_i=0 \ldots\rangle \underset{\substack{\uparrow \\ \text{assumption}}}{=} \frac{a_i^+}{\sqrt{m_i+1}} |\ldots n_i=m_i \ldots\rangle$$

$$= \frac{1}{\sqrt{m_i+1}} \sqrt{m_i+1}\, |\ldots n_i=m_i+1 \ldots\rangle$$

$$= |\ldots n_i=m_i+1 \ldots\rangle$$

Applying a product of (commuting) operators like this gives

$$\prod_i \frac{(a_i^+)^{m_i}}{\sqrt{m_i!}} |0\rangle = |n_1=m_1, n_2=m_2, \ldots\rangle$$

or in simpler notation

$$\prod_i \frac{(a_i^+)^{m_i}}{\sqrt{m_i!}} |0\rangle = |m_1, m_2, \ldots\rangle$$