Solutions for the questions on Grassmannians in the end of the lecture notes

① The question asks to evaluate the integral

$$I = \int \prod_{j=1}^{3} (d\eta_j^* \, d\eta_j) \, (\eta_1^* + \eta_2^* + \eta_3^*)^2 (\eta_1 + \eta_2 + \eta_3)^2$$
$$\exp(2i\eta_1^* \eta_2 - 2i\eta_2^* \eta_1 + \eta_3^* \eta_3)$$

The integrand involves squares, and these vanish due to the properties of the Grassmannians. In particular we have

$$(\eta_1 + \eta_2 + \eta_3)^2 = \underset{\overset{\shortparallel}{0}}{\eta_1^2} + \underset{\overset{\shortparallel}{0}}{\eta_2^2} + \underset{\overset{\shortparallel}{0}}{\eta_3^2} + \underbrace{(\eta_1 \eta_2 + \eta_2 \eta_1)}_{=0}$$
$$+ \underbrace{(\eta_1 \eta_3 + \eta_3 \eta_1)}_{=0} + \underbrace{(\eta_2 \eta_3 + \eta_3 \eta_2)}_{=0}$$
$$= 0$$

Hence $I = 0$.

If the integrand were nonzero we would have started by defining

$$2i\eta_1^* \eta_2 - 2i\eta_2^* \eta_1 + \eta_3^* \eta_3 = -\underline{\eta}^+ A \underline{\eta} \;, \quad A = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\langle \ldots \rangle = \frac{1}{c} \int \prod_{j=1}^{3} (d\eta_j^* \, d\eta_j) \, e^{-\underline{\eta}^+ A \underline{\eta}} \ldots$$

$$c = \int \prod_{j=1}^{3} (d\eta_j^* \, d\eta_j) \, e^{-\underline{\eta}^+ A \underline{\eta}} \ldots$$
$$= \det A = -2i(-2i)(-1) = 4$$

Then we would have

$$I = \underset{\overset{\shortparallel}{4}}{c} \langle (\eta_1^* + \eta_2^* + \eta_3^*)^2 (\eta_1 + \eta_2 + \eta_3)^2 \rangle$$

which could have been evaluated using Wick's thm.

(2) We have

$$\operatorname{tr}(\lambda I - A)^{-1}$$
$$= \frac{\partial}{\partial\lambda} \operatorname{tr} \ln(\lambda I - A)$$

With $\det = \exp \operatorname{tr} \ln$ and thus $\ln\det = \operatorname{tr}\ln$ this turns into

$$\operatorname{tr}(\lambda I - A)^{-1}$$
$$= \frac{\partial}{\partial\lambda} \ln\det(\lambda I - A)$$
$$= \frac{\frac{\partial}{\partial\lambda}\det(\lambda I - A)}{\det(\lambda I - A)}$$
$$= \frac{\partial}{\partial\mu} \left.\frac{\det(\mu I - A)}{\det(\lambda I - A)}\right|_{\mu=\lambda}$$

If $\lambda I - A$ is positive definite we can use

$$\frac{1}{\det(\lambda I - A)} = \int d[\underline{z}]\; e^{-\underline{z}^{+}(\lambda I - A)\underline{z}}$$

$d[\underline{z}] = dz_1^* dz_1 \dots dz_N^* dz_N$ where A is an $N \times N$ matrix and $z_i \in \mathbb{C}$

$$\det(\mu I - A) = \int d[\underline{\eta}]\; e^{-\underline{\eta}^{+}(\mu I - A)\underline{\eta}}$$

$d[\underline{\eta}] = d\eta_1^* d\eta_1 \dots d\eta_N^* d\eta_N$ where η_i are Grassmannians

This leads to

$$\operatorname{tr}(\lambda I - A)^{-1} = \frac{\partial}{\partial\mu} \int d[\underline{z}]\, d[\underline{\eta}]\; e^{-\underline{z}^{+}(\lambda I - A)\underline{z} - \underline{\eta}^{+}(\mu I - A)\underline{\eta}}\Big|_{\mu=\lambda}$$

If $\lambda I - A$ is negative definite we can use

$$\frac{1}{\det(\lambda I - A)} = (-1)^N \frac{1}{\det(A - \lambda I)} = (-1)^N \int d[\underline{z}]\, e^{-\underline{z}^{+}(A - \lambda I)\underline{z}}$$

$$\det(\mu I - A) = (-1)^N \det(A - \mu I) = (-1)^N \int d[\underline{\eta}]\, e^{-\underline{\eta}^{+}(A - \mu I)\underline{\eta}}$$

This leads to

$$\operatorname{tr}(\lambda I-A)^{-1} = \frac{\partial}{\partial\mu}\int d[\underline{z}]\, d[\underline{\eta}]\, e^{-\underline{z}^{+}(A-\lambda I)\underline{z}-\underline{\eta}^{+}(A-\mu I)\underline{\eta}}\Big|_{\mu=\lambda}$$

Slightly different but equivalent formulas are obtained if one uses in the beginning

$$\begin{aligned}\operatorname{tr}(\lambda I-A)^{-1} &= \frac{\partial}{\partial\lambda}\ln\det(\lambda I-A)\\ &= -\frac{\partial}{\partial\lambda}\ln\det(\lambda I-A)^{-1}\\ &= -\det(\lambda I-A)\frac{\partial}{\partial\lambda}\det(\lambda I-A)^{-1}\end{aligned}$$